Weakly compact approximation in Banach spaces

Edward Odell* and Hans-Olav Tylli**

Abstract. *The Banach space E has the weakly compact approximation property (W.A.P. for short) if there is a constant $C < \infty$ so that for any weakly compact set $D \subset E$ and $\varepsilon > 0$ there is a weakly compact operator $V : E \to E$ satisfying* $\sup_{x \in D} \|x - Vx\| < \varepsilon$ *and $\|V\| \leq C$. We give several examples of Banach spaces both with and without this approximation property. Our main results demonstrate that the James-type spaces from a general class of quasi-reflexive spaces (which contains the classical James' space J) have the W.A.P, but that James' tree space JT fails to have the W.A.P. It is also shown that the dual J^* has the W.A.P. It follows that the Banach algebras $W(J)$ and $W(J^*)$, consisting of the weakly compact operators, have bounded left approximate identities. Among the other results we obtain a concrete Banach space Y so that Y fails to have the W.A.P., but Y has this approximation property without the uniform bound C.*

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1. **Introduction.**

A Banach space E is said to have the *weakly compact approximation property* (abbreviated W.A.P.) if there is a constant $C < \infty$ such that for any weakly compact set $D \subset E$ and $\varepsilon > 0$ there is a weakly compact operator $V : E \to E$ satisfying

$$\sup_{x \in D} \|x - Vx\| < \varepsilon \quad \text{and} \quad \|V\| \leq C. \tag{1.1}$$

This (bounded) weakly compact approximation property was introduced by Astala and Tylli [AT]. The applications mentioned below were the principal motivation for this in [AT], but the W.A.P. is a natural notion worthy of study in its own right. Clearly any reflexive Banach space has the W.A.P., but this property is quite rare for non-reflexive spaces. For instance, if E is a $\mathcal{L}^1$- or $\mathcal{L}^\infty$-space, then E has the W.A.P. if and only if E has the Schur property, see [AT,Cor.3]. We note that a different notion is obtained by considering the uniform approximation of the identity operator on compact sets by weakly compact operators (see e.g. Reinov [R], Grønbæk and Willis [GW], and Lima, Nygaard and Oja [LNO] for this).

The weakly compact approximation property defined by (1.1) has some unexpected applications. The key fact [AT,Thm. 1] here is that the Banach space F has the W.A.P. if and only if the measure of weak non-compactness

$$\omega(S) = \inf\{\varepsilon > 0 : SB_E \subset D + \varepsilon B_F, D \subset F \text{ weakly compact}\}$$

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and the weak essential norm $S \mapsto \|S\|_w \equiv \mathrm{dist}(S, W(E,F))$ are uniformly comparable in the space $L(E,F)$ of bounded linear operators $E \to F$ for all Banach spaces E. Here $B_E = \{x \in E : \|x\| \leq 1\}$ and $W(E,F)$ stands for the weakly compact operators $E \to F$. The fact that c_0 fails to have the W.A.P. was then applied in [AT,Thm. 4 and Cor. 5] to show that $\omega(S)$ is in general neither uniformly comparable to $\omega(S^*)$ nor to $\omega(JS)$ for arbitrary linear into isometries J. Subsequently, weakly compact approximation properties were exploited in [T2] to obtain examples of Banach spaces E and F, where $\|S\|_w$ is not uniformly comparable to $\|S^*\|_w$. Further applications of the W.A.P. arise from the fact that the Banach algebra $W(E)$ has a bounded left approximate identity whenever E has the W.A.P.

This paper contains several results and examples about the weakly compact approximation property for Banach spaces. A first natural question is in which sense "almost reflexive" Banach spaces still possess the W.A.P. A principal aim is to discuss the W.A.P. for the class of quasi-reflexive Banach spaces E, where $\dim(E^{**}/E) < \infty$. In sections 2 and 3 we show that the classical James space J and its dual J^* have the W.A.P. These results imply that $W(J)$ and $W(J^*)$ have bounded left approximate identities. In this direction Loy and Willis [LW] established that $W(J)$ has a bounded right approximate identity. In section 4 we extend the results of section 2 by proving that the quasi-reflexive James-like spaces constructed by Bellenot, Haydon and Odell in [BHO] have the W.A.P. These positive results are further highlighted by the recent discovery of Argyros and Tolias (see [ArT,Prop. 14.10]) that there exist quasi-reflexive hereditarily indecomposable Banach spaces E that do not have the W.A.P.

In section 5 we present a permanence property for weakly compact approximation properties, which implies among other things that certain vector-valued sequence spaces, including $\ell^1(\ell^p)$ and $\ell^p(\ell^1)$ for $1 < p < \infty$, have the W.A.P.

Section 6 contains a number of additional examples of spaces failing the W.A.P. For instance, we show that James' tree space JT does not have the W.A.P. We also establish that the W.A.P. differs from the corresponding "unbounded" W.A.P. (where the uniform bound $\|V\| \leq C$ is removed from (1.1)). Moreover, we obtain a concrete Banach space Y so that the quotient Y^{**}/Y is isometric to ℓ^2, but Y does not have the W.A.P. (This example yields a simpler negative answer to a question from [AT] than the quasi-reflexive spaces constructed in [ArT]). Another natural problem is whether E always has the W.A.P. if E is ℓ^1-saturated and E has a Schauder basis. Indeed this is false, since as we show the Lorentz sequence spaces $d(w,1)$, as well as the Azimi-Hagler spaces from [AH], do not have the W.A.P.

The basic terminology and notation related to Banach spaces will follow [LT].

2. **The James space J has the W.A.P.**

The most well-known (and first discovered) quasi-reflexive Banach space J was introduced by James [J1]. The fact that J has the W.A.P. follows from the more general results of section 4. However, that argument is more complicated and many of the ideas we use there, and in section 3, are well illustrated by first presenting them for J.

Recall that a real-valued sequence $x = (x_j) \in J$ if $\lim_{j\to\infty} x_j = 0$ and the square

variation norm

$$\|x\|^2 = \sup \sum_{j=1}^{n} |x_{p_{j+1}} - x_{p_j}|^2 < \infty, \tag{2.1}$$

where the supremum is taken over all indices $1 \le p_1 < p_2 < \ldots < p_n < p_{n+1}$ and $n \in \mathbf{N}$. The monograph [FG] is a convenient source of results (as well as further references) about J. Recall that the coordinate basis (e_n) is a shrinking Schauder basis for J, so that J^{**} can be identified with the set of scalar sequences $x = (x_j)$ for which $\sup_n \| \sum_{j=1}^{n} a_j e_j \| < \infty$. Moreover, $J^{**} = \{x + \lambda \mathbf{1} : x \in J, \lambda \in \mathbf{R}\}$, where $\mathbf{1} = (1, 1, 1, \ldots)$.

The question whether (1.1) is satisfied for any weakly compact subset $D \subset J$ can be viewed as a concrete approximation problem for J that may have independent interest. The set $\{e_n : n \in \mathbf{N}\} \cup \{0\}$ of J is already a non-trivial test for (1.1), since the sequence (e_n) is weakly null in J. It turns out that (somewhat surprisingly) the desired approximating operators $V \in W(J)$ are perturbations of the identity operator by certain double averaging functionals over consecutive blocks.

We first state a well known general auxiliary result. It is convenient to put $[n, m) = \{n, \ldots, m-1\}$ if $m, n \in \mathbf{N}$ and $m > n$.

Lemma 2.1. Suppose that E is a Banach space with a normalized Schauder basis (e_n), and let $D \subset E$ be an arbitrary weakly compact subset. Then for any $\delta > 0$ and $n \in \mathbf{N}$ there is $m > n$ such that for any $x = \sum_{j=1}^{\infty} a_j e_j \in D$ there is an index $j = j(x) \in [n, m)$ satisfying $|a_j| < \delta$.

Proof. Suppose to the contrary that there is $\delta > 0$ and $n \in \mathbf{N}$ so that for any $m > n$ there is an element $x_m = \sum_{j=1}^{\infty} a_j^{(m)} e_j \in D$ satisfying $|a_j^{(m)}| \ge \delta$ for all $j \in [n, m)$. By the weak compactness of D we may assume that $x_m \xrightarrow{w} x = \sum_{j=1}^{\infty} a_j e_j \in D$ as $m \to \infty$. Since $a_j = \lim_{m\to\infty} a_j^{(m)}$ for $j \in \mathbf{N}$, we arrive at the contradiction that $|a_j| \ge \delta$ for each $j \ge n$. $\square$

Theorem 2.2. James' space J has the W.A.P.

Proof. The argument will be split into several steps. Let $D \subset J$ be a fixed weakly compact subset and $\varepsilon > 0$. By homogeneity there is no loss of generality to assume that $D \subset B_J$.

Step 1. We start by fixing some notation. Given natural numbers $1 \le n_1 < \ldots < n_{k+1}$ we introduce the related averaging functionals $A_{[n_j, n_{j+1})}$ and $A_{(n_1, \ldots, n_{k+1})}$ on J by

$$A_{[n_j,n_{j+1})}(x) = \frac{1}{n_{j+1} - n_j} \sum_{s=n_j}^{n_{j+1}-1} a_s, \quad A_{(n_1,\ldots,n_{k+1})}(x) = \frac{1}{k} \sum_{j=1}^{k} A_{[n_j,n_{j+1})}(x),$$

for $x = \sum_{i=1}^{\infty} a_i e_i \in J$ and $j = 1, \ldots, k$. Note that $\|A_{[n_j,n_{j+1})}\| \le 1$ for each j, since

$$|A_{[n_j,n_{j+1})}(x)| = \frac{1}{n_{j+1} - n_j} |\langle \sum_{s=n_j}^{n_{j+1}-1} e_s^*, x \rangle| \le \|x\|.$$

Here $(e_s^*) \subset J^*$ stands for the sequence of biorthogonal functionals to (e_s); clearly $\|e_s^*\| = 1$ for $s \in \mathbf{N}$. We next show that certain double averages $A_{(n_1,\ldots,n_{k+1})}$ are uniformly small on the weakly compact set D, provided k is large enough.

Claim 1. Let $\delta > 0$ and $n = n_1 \in \mathbf{N}$ be arbitrary. Then there is $k \in \mathbf{N}$ and natural numbers $n_1 < \ldots < n_{k+1}$ so that

$$|A_{(n_1,\ldots,n_{k+1})}(x)| < 3\delta \quad \text{for all } x \in D.$$

Proof of Claim 1. Use Lemma 2.1 repeatedly to choose a sequence $n = n_1 < n_2 < \ldots$ in $\mathbf{N}$, such that for any $j \in \mathbf{N}$ and $x = \sum_{i=1}^{\infty} a_i e_i \in D$ there is an index $i = i(x) \in [n_j, n_{j+1})$ satisfying $|a_i| < \delta$. Let $k \in \mathbf{N}$ be given. For each fixed $x = \sum_{i=1}^{\infty} a_i e_i \in D$ put

$$I = \{j \le k : \text{there is } i \in [n_j, n_{j+1}) \text{ with } |a_i| \ge 2\delta\}$$

(note that I depends on x, k and δ). In order to choose k suppose that $I = \{j_1, \ldots, j_r\}$ and pick $p_s, q_s \in [n_{j_s}, n_{j_s+1})$ such that $|a_{p_s}| < \delta$ and $|a_{q_s}| \ge 2\delta$ for $s = 1, \ldots, r$. The square variation norm (2.1) satisfies

$$\|x\| \ge (\sum_{s=1}^{r} |a_{q_s} - a_{p_s}|^2)^{1/2} \ge |I|^{1/2}\delta.$$

We deduce that the cardinality $|I| \le \frac{1}{\delta^2}$, because $D \subset B_J$ by assumption. Note further that $|A_{[n_j,n_{j+1})}(x)| < 2\delta$ whenever $j \notin I$, since $|a_s| < 2\delta$ for all $s \in [n_j, n_{j+1})$ in this event. By putting these estimates together we get that

$$|A_{(n_1,\ldots,n_{k+1})}(x)| \le \frac{1}{k}(|I| + (k - |I|)2\delta) \le \frac{1}{k\delta^2} + 2\delta < 3\delta$$

once we pick $k > \delta^{-3}$. The completes the argument for Claim 1.

Step 2. Fix a decreasing null-sequence (ε_j) such that $\sum_{j=1}^{\infty} \varepsilon_j < \varepsilon/\sqrt{2}$. By successive applications of Claim 1 we find a sequence of consecutive subdivisions $1 = n_{p_1} < n_{p_1+1} < \ldots < n_{p_2} < n_{p_2+1} < \ldots < n_{p_3} < n_{p_3+1} < \ldots$ of $\mathbf{N}$ such that

$$|A_j(x)| < \varepsilon_j \quad \text{for all } x \in D \text{ and } j \in \mathbf{N}, \tag{2.2}$$

where we set $A_j = A_{(n_{p_j}, n_{p_j+1}, \ldots, n_{p_{j+1}})}$ for $j \in \mathbf{N}$. Let $I_j = [n_{p_j}, n_{p_{j+1}})$ for $j \in \mathbf{N}$, which is the "support" (with respect to the coordinate basis (e_s)) of the functional A_j on J. Put $g_j = \sum_{i \in I_j} e_i$, so that $\|g_j\| \le \sqrt{2}$ for $j \in \mathbf{N}$. Define the linear map V on J by

$$Vx = x - \sum_{j=1}^{\infty} A_j(x) g_j, \quad x \in J. \tag{2.3}$$

We verify in three separate steps that V provides a uniformly bounded weakly compact approximating operator for the given weakly compact set $D \subset B_J$ as required by (1.1).

Claim 2. $\|x - Vx\| < \varepsilon$ *for all* $x \in D$.

Proof of Claim 2. It follows from (2.3) and the choice of (ε_j) that

$$\|x - Vx\| = \|\sum_{j=1}^{\infty} A_j(x) g_j\| \le \sum_{j=1}^{\infty} |A_j(x)| \cdot \|g_j\| < \sqrt{2} \cdot \sum_{j=1}^{\infty} \varepsilon_j < \varepsilon.$$

Claim 3. $\|V\| \le 3$ *(independently of the subdivisions).*

Proof of Claim 3. Put $\tilde{V}x = \sum_{j=1}^{\infty} A_j(x) g_j$ for $x \in J$. It suffices to verify that $\|\tilde{V}\| \le 2$.

It is convenient to denote $y = \sum_{j=1}^{\infty} y(j) e_j \in J$ in the argument. Assume that $x \in J$ is finitely supported, and suppose that $q_1 < \ldots < q_{n+1}$ is a sequence of coordinates that realizes the square variation norm (2.1) of $\tilde{V}x$. We first split

$$\begin{aligned} \|\tilde{V}x\| &= (\sum_{i=1}^{n} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2} \\ &\le (\sum_{i \in A} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2} + (\sum_{i \in B} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2}, \end{aligned} \tag{2.4}$$

where $i \in A$ if both $q_i, q_{i+1} \in I_j$ for some j, while $i \in B$ if q_i and q_{i+1} belong to different intervals. Note that if $i \in A$ and $q_i, q_{i+1} \in I_j$, then $|\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)| = |A_j(x) - A_j(x)| = 0$ by definition, so that the term $(\sum_{i \in A} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2}$ actually vanishes. To estimate the second term in (2.4) we split

$$\begin{aligned} (\sum_{i \in B} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2} \le &(\sum_{i \in B_1} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2} + \\ &+ (\sum_{i \in B_2} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2}. \end{aligned} \tag{2.5}$$

In (2.5) the set B_1 contains every $2k+1$:th term of B, and B_2 the remaining ones. Consider a single term $|\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|$ for some $i \in B_1$, and suppose that $q_i \in I_j$ and $q_{i+1} \in I_k$ (where $j < k$). In this case

$$|\tilde{V}x(q_i) - \tilde{V}x(q_{i+1})| = |A_j(x) - A_k(x)|.$$

By definition the double average $A_j(x)$ is a convex combination of $\{x(s) : s \in I_j\}$ (and analogously for $A_k(x)$). Consequently there are indices $r_i \in I_j$ and $r_{i+1} \in I_k$ so that $|A_j(x) - A_k(x)| \le |x(r_i) - x(r_{i+1})|$. Since B_1 contains every second index from B, it is easy to check that the corresponding sequence (r_i) is increasing, so we obtain that $(\sum_{i \in B_1} |A_j(x) - A_k(x)|^2)^{1/2} \le \|x\|$. By arguing in a similar manner for the sum over the "even" indices $i \in B_2$, we get from (2.5) that

$$\|\tilde{V}x\| = (\sum_{i \in B} |\tilde{V}x(q_{i+1}) - \tilde{V}x(q_i)|^2)^{1/2} \le 2\|x\|.$$

Hence we get by approximating that $\|\tilde{V}x\| \le 2\|x\|$ for all $x \in J$. This establishes Claim 3.

Claim 4. $V \in W(J)$.

Proof of Claim 4. Recall that $V \in W(J)$ if and only if $V^{**}(\mathbf{1}) \in J$, where $\mathbf{1} = (1,1,\ldots) \in J^{**} \setminus J$. Put $f_m = \sum_{j=1}^m e_j$ for $m \in \mathbf{N}$. Note that $f_{m_k} \xrightarrow{w^*} \mathbf{1}$ and $Vf_{m_k} \xrightarrow{w^*} V^{**}\mathbf{1}$ in J^{**} as $k \to \infty$ for each subsequence (f_{m_k}) of (f_m). Consider $s_k = \sum_{j=1}^k g_j$ for $k \in \mathbf{N}$, which determines a subsequence of (f_m). Here

$$Vg_j = g_j - \sum_{k=1}^{\infty} A_k(g_j)g_k = 0, \quad j \in \mathbf{N},$$

since the averages $A_k(g_j) = \delta_{j,k}$ for $j,k \in \mathbf{N}$. It follows that $Vs_k = 0$ for $k \in \mathbf{N}$, so that $V^{**}\mathbf{1} = 0$. Thus $V \in W(J)$. This completes the proof of Theorem 2.2. $\square$

Remarks 2.3. (i) Simpler uniformly bounded approximating operators $V \in W(J)$ are available for the particular weakly compact set $D = \{e_n : n \in \mathbf{N}\} \cup \{0\} \subset J$. Indeed, define $V_k \in W(J)$ for $k \in \mathbf{N}$ by

$$V_k x = x - \sum_{j=1}^{\infty} A_{[kj,k(j+1))}(x)h_j, \quad \text{where } h_j = \sum_{s=kj}^{k(j+1)-1} e_s, \ j \in \mathbf{N}.$$

Then V_k satisfies (1.1) for D and $\varepsilon > 0$ once $\frac{\sqrt{2}}{k} < \varepsilon$, since $\frac{1}{|I|}|\sum_{s\in I} x(s)| \le \frac{1}{|I|}$ for intervals $I \subset \mathbf{N}$ and $x = \sum_{s=1}^{\infty} x(s)e_s \in D$. The uniform bound for $\|V_k\|$ and the weak compactness of V_k are easy modifications of Claims 3 and 4 above.

Let $S_k \in L(J)$ be the forward k-shift on J for $k = 0,1,2,\ldots$. The reader may also wish to check that $V_m = I - \frac{1}{m+1}\sum_{k=0}^m S_k \in W(J)$ for $m \in \mathbf{N}$, and that V_m satisfies (1.1) for D and $\varepsilon > 0$ once m is large enough.

(ii) The argument of section 4 yields more complicated weakly compact approximating operators when applied to J (e.g. the double averaging functionals are replaced by less explicit convex combinations, and there is a "shift"-like perturbation of the identity).

The following vector-valued analogue of James' space J has been studied in several contexts, see e.g. [PQ] and [P]. Let E be a Banach space and $1 < p < \infty$. The sequence $x = (x_j) \subset E$ belongs to $J_p(E)$ if $\lim_{j\to\infty} x_j = 0$ and the p-variation norm

$$\|x\|^p = \sup_{n;p_1<\ldots<p_{n+1}} \sum_{j=1}^n \|x_{p_{j+1}} - x_{p_j}\|^p < \infty.$$

Here $J_p(E)^{**}/J_p(E) \approx E$ for reflexive spaces E, see e.g. [W,Cor. 2]. Straightforward modifications of the argument in Theorem 2.2 yield that $J_p = J_p(\mathbf{R})$ also has the W.A.P. for $1 < p < \infty$. More generally, Theorem 5.3 below implies that $J_p(\mathbf{R}^n) \approx J_p \oplus \ldots \oplus J_p$ (n summands) has the W.A.P. for $n \in \mathbf{N}$. This suggests the following problem (a similar question may obviously be raised for other James-Lindenstrauss type constructions).

Problem 2.4. Does $J_p(E)$ have the W.A.P. whenever E is reflexive and $1 < p < \infty$?

Recall that a Banach algebra A has a *bounded left approximate identity* (abbreviated B.L.A.I.) if there is a bounded net $(x_\alpha) \subset A$ such that

$$\lim_\alpha \|y - x_\alpha y\| = 0 \quad \text{for } y \in A. \tag{2.6}$$

A *bounded right approximate identity* (B.R.A.I. for short) in A is obtained by considering $\|y - yx_\alpha\|$ in (2.6). The following observation contains an application of the W.A.P. to algebras of weakly compact operators.

Proposition 2.5. (i) If E has the W.A.P., then the Banach algebra $W(E)$ has a B.L.A.I.

(ii) $W(J)$ has a B.L.A.I.

Proof. (i) Let $U \in W(E)$ and $\varepsilon > 0$ be arbitrary. By applying (1.1) to the weakly compact set $\overline{UB_E}$ we obtain $V \in W(E)$ satisfying $\|V\| \leq C$ and

$$\|U - VU\| = \sup_{x \in B_E} \|Ux - VUx\| < \varepsilon.$$

Here $C < \infty$ is a uniform constant. It follows that $W(E)$ has a B.L.A.I. by a well-known sufficient condition from [BD,Prop. 11.2].

Part (ii) follows from (i) and Theorem 2.2. □

Remarks 2.6. (i) Loy and Willis [LW,Cor. 2.4] established that $W(J)$ has a B.R.A.I. using a different matrix-type argument. Their result suggested the problem whether $W(J)$ also has a B.L.A.I. This question was not stated in print, but it was known to specialists (R.J. Loy and G.A. Willis, personal communication).

Several authors have studied the existence of bounded left or right approximate identities for algebras of operators, chiefly for closed subalgebras of the compact operators on a Banach space. For such subalgebras the existence of a B.R.A.I. implies the existence of a B.L.A.I. [GW,Cor. 2.7], but no results of this type are known in the setting of $W(E)$. We refer to [D,2.9.37 and 2.9.67] for further information and references.

(ii) The converse of Proposition 2.5.(i) fails already for $E = \ell^1$, see [T1, p. 107].

3. J^* has the W.A.P.

Recall that the dual J^* is also quasi-reflexive of order 1, but that J^* is quite different from J as a Banach space. For instance, the norm in J^* is not given by any concrete formula, J^* does not embed into J by [J2,Thm. 3] and J does not embed into J^* by [A,Thm. 7] (see also [P] for a different approach). This provides ample motivation for considering weakly compact approximation in J^*.

The argument that J^* has the W.A.P. follows the basic outline of section 2, but the details are more involved. We recall a few relevant facts about J^*. Put $f_n = \sum_{j=1}^n e_j \in J$ for $n \in \mathbf{N}$. Then (f_n) is a boundedly complete Schauder basis for J. Put $S^*(x) =$

$\sum_{s\in S} x(s)$ for $x = \sum_{s=1}^\infty x(s)f_s \in J$ and any interval $S \subset \mathbf{N}$. In the basis (f_n) the square variation norm (2.1) becomes

$$\|x\| = \sup_{n;\ S_1<\ldots<S_n} (\sum_{j=1}^n S_j^*(x)^2)^{1/2}, \quad x = \sum_{s=1}^\infty x(s)f_s \in J, \tag{3.1}$$

where $S_1,\ldots,S_n$ are intervals of $\mathbf{N}$ satisfying $\max S_i < \min S_{i+1}$ for $i = 1,\ldots,n-1$ (the interval S_n may be unbounded). We denote this by $S_1 < S_2 < \ldots < S_n$. Thus $\|S^*\| = 1$ whenever $S \subset \mathbf{N}$ is an interval. We will require the fact that

$$\|\sum_{k=1}^n c_k S_k^*\| \le (\sum_{k=1}^n |c_k|^2)^{1/2} \tag{3.2}$$

whenever $S_1 < S_2 < \ldots < S_n$ are intervals of $\mathbf{N}$ and $c_1,\ldots,c_n$ are scalars. Indeed,

$$|\langle\sum_{k=1}^n c_k S_k^*, x\rangle| = |\sum_{k=1}^n c_k S_k^*(x)| \le (\sum_{k=1}^n |c_k|^2)^{1/2}(\sum_{k=1}^n S_k^*(x)^2)^{1/2} \le (\sum_{k=1}^n |c_k|^2)^{1/2}\|x\|$$

for $x = \sum_{s=1}^\infty x(s)f_s \in J$.

The sequence (f_n^*) of biorthogonal functionals to (f_n) forms a w^*-basis for J^*, that is, for any $x^* \in J^*$ there is a unique scalar sequence $(a_j) = (x^*(e_j))$ so that $x^* = (w^*)\sum_{j=1}^\infty a_j f_j^*$ as a w^*-convergent sum in J^*. It is known that the limit $\lim_{j\to\infty} a_j$ exists for this w^*-representation of x^*. We recall the argument, since we will actually need the more precise quantitative version given below in part (ii).

Lemma 3.1. Let $x^* = (w^*)\sum_{j=1}^\infty a_j f_j^*$ be the unique w^*-convergent representation of $x^* \in B_{J^*}$.

(i) Then $\lim_{j\to\infty} a_j$ exists.

(ii) There is a uniform constant $C < \infty$ with the following property: Let $\varepsilon > 0$ and suppose that for some $k \in \mathbf{N}$ there are indices $p_1 < q_1 < p_2 < q_2 < \ldots < p_k < q_k$ with $|a_{p_i} - a_{q_i}| > \varepsilon$ for $i = 1,\ldots,k$. Then $k \le \frac{C}{\varepsilon^2}$.

Proof. Suppose that there are $\varepsilon > 0$ and indices $p_1 < q_1 < p_2 < q_2 < \ldots < p_k < q_k$ so that $|a_{p_j} - a_{q_j}| > \varepsilon$ for all $j \le k$. By a result of Casazza, Lin and Lohman (see [CLL,thm. 16] or [FG,Thms. 2.d.1 and 2.c.9]) the sequence $(f_{q_j} - f_{p_j})_{j=1}^k$ is equivalent to the unit vector basis of ℓ_k^2 with uniform isomorphism constants independent of (p_j), (q_j) and $k \in \mathbf{N}$. Fix k and consider

$$x_k = \frac{1}{\sqrt{k}}\sum_{j=1}^k \theta_j(f_{q_j} - f_{p_j}) \in J,$$

where the signs $\theta_1,\ldots,\theta_k$ are chosen so that $\theta_j(a_{q_j} - a_{p_j}) = |a_{p_j} - a_{q_j}|$ for $j = 1,\ldots,k$. Thus $\|x_k\| \le C$, where C is a uniform constant. We get that

$$C \ge x^*(x_k) = \frac{1}{\sqrt{k}}\sum_{j=1}^k \theta_j\langle x^*, f_{q_j} - f_{p_j}\rangle = \frac{1}{\sqrt{k}}\sum_{j=1}^k \theta_j(a_{q_j} - a_{p_j}) \ge \varepsilon\sqrt{k},$$

which proves both (i) and (ii). $\square$

Suppose that $x^* = (w^*)\sum_{j=1}^\infty a_j f_j^* \in J^*$ and let $a = \lim_j a_j$. Put $S_\infty = \mathbf{N}$, so that $S_\infty^*(\sum_{j=1}^\infty c_j f_j) = \sum_j c_j$ for $\sum_{j=1}^\infty c_j f_j \in J$. We can write

$$x^* = \sum_{j=1}^\infty b_j f_j^* + aS_\infty^*, \tag{3.3}$$

where $b_j \equiv a_j - a \to 0$ as $j \to \infty$. In (3.3) the sum $\sum_{j=1}^\infty b_j f_j^*$ is norm-convergent in J^*. We will need the following variant of Lemma 2.1 for weakly compact subsets of J^*.

Lemma 3.2. Let $D \subset J^*$ be a weakly compact set.

(i) Suppose that $(x_n^*) \subset D$ is a sequence, where $x_n^* = \sum_{j=1}^\infty a_j^{(n)} f_j^* + a_\infty^{(n)} S_\infty^*$ is written as in (3.3) for $n \in \mathbf{N}$. Then there is a subsequence of (x_n^*), still denoted by (x_n^*), so that

$$x_n^* \xrightarrow{w} x^* = \sum_{j=1}^\infty a_j f_j^* + a_\infty S_\infty^* \quad \text{as } n \to \infty,$$

where x^* is represented as in (3.3) and $\lim_{n\to\infty} a_j^{(n)} = a_j$ for all $j \in \mathbf{N} \cup \{\infty\}$.

(ii) For all $n \in \mathbf{N}$ and $\delta > 0$ there is $m > n$ so that for all $x^* = \sum_{j=1}^\infty a_j f_j^* + a_\infty S_\infty^* \in D$ in the representation (3.3), there is $j = j(x^*) \in [n, m)$ satisfying $|a_j| < \delta$.

Proof. (i) The weak compactness of D gives a subsequence of (x_n^*), still denoted by (x_n^*), so that $x_n^* \xrightarrow{w} x^*$ as $n \to \infty$. Write $x^* = \sum_{j=1}^\infty a_j f_j^* + a_\infty S_\infty^*$ as in (3.3). Let $x^{**} \in J^{**}$ satisfy $x^{**}(S_\infty^*) = 1$ and $x^{**}(f_j^*) = 0$ for all $j \in \mathbf{N}$. Hence $a_\infty^{(n)} = x^{**}(x_n^*) \to x^{**}(x^*) = a_\infty$ as $n \to \infty$. Moreover, $a_j^{(n)} + a_\infty^{(n)} = x_n^*(f_j) \to x^*(f_j) = a_j + a_\infty$ as $n \to \infty$ for $j \in \mathbf{N}$. It follows that $\lim_{n\to\infty} a_j^{(n)} = a_j$ for each $j \in \mathbf{N}$.

(ii) Suppose to the contrary that $n \in \mathbf{N}$ and $\delta > 0$ are such that for any $m > n$ there is $x_m^* = \sum_{j=1}^\infty a_j^{(m)} f_j^* + a_\infty^{(m)} S_\infty^* \in D$ represented as in (3.3), for which $|a_j^{(m)}| \geq \delta$, for all $j \in [n, m)$. By part (i) there is a subsequence (x_m^*) for which

$$x_m^* \xrightarrow{w} x^* = \sum_{j=1}^\infty a_j f_j^* + a_\infty S_\infty^* \quad \text{as } m \to \infty,$$

where x^* is written as in (3.3) and $\lim_{m\to\infty} a_j^{(m)} = a_j$ for all $j \in \mathbf{N}$. This implies that $|a_j| \geq \delta$ for all $j \geq n$, which contradicts the properties of the expansion (3.3). $\square$

We are ready to prove the main result of this section.

Theorem 3.3. *J^* has the W.A.P.*

Proof. Let $D \subset B_{J^*}$ be a fixed weakly compact subset and $\varepsilon > 0$. The desired approximating operator $V \in W(J^*)$ satisfying (1.1) will again be constructed in several stages.

Step 1. We first fix some notation. Given $n_1 < \ldots < n_k < n_{k+1}$ we define the averaging functionals $B_{[n_j,n_{j+1})}$ and $B_{(n_1,\ldots,n_{k+1})}$ on J^* by

$$B_{[n_j,n_{j+1})}(x^*) = \frac{1}{n_{j+1}-n_j}\sum_{s=n_j}^{n_{j+1}-1} a_s, \quad B_{(n_1,\ldots,n_{k+1})}(x^*) = \frac{1}{k}\sum_{j=1}^{k} B_{[n_j,n_{j+1})}(x^*)$$

for $x^* = \sum_{i=1}^\infty a_i f_i^* + a_\infty S_\infty^* \in J^*$ written as in (3.3) and $j = 1,\ldots,k$. Note that $B_{[n_j,n_{j+1})} \in J^{**}$ and that $\|B_{[n_j,n_{j+1})}\| \le 2$ for $j \in \mathbf{N}$. In fact, it is easy to see that $|a_s| = |x^*(f_s)| + |a_\infty| \le 2\|x^*\|$ for $x^* = \sum_{i=1}^\infty a_i f_i^* + a_\infty S_\infty^* \in J^*$ and $s \in \mathbf{N}$.

Let $n \in \mathbf{N}$ and $\delta > 0$ be given. By applying Lemma 3.2.(ii) repeatedly for $\delta/2 > 0$ we find a sequence $n = n_1 < n_2 < \ldots$ of $\mathbf{N}$, such that for any $x^* = \sum_{i=1}^\infty a_i f_i^* + a_\infty S_\infty^* \in D$ and $j \in \mathbf{N}$ there is $i = i(x^*) \in [n_j, n_{j+1})$ satisfying $|a_i| < \delta/2$. Here $n_{j+1} = m(n_j, \delta/2)$ is given by Lemma 3.2.(ii). We first verify that sufficiently long double averages $B_{(n_1,\ldots,n_{k+1})}$ are uniformly small on the weakly compact set D.

Claim 1. There is $k \in \mathbf{N}$ such that

$$|B_{(n_1,\ldots,n_{k+1})}(x^*)| < 2\delta \quad \text{for all } x^* \in D.$$

Proof. The idea resembles that of Claim 1 in Theorem 2.2. Let $x^* \in D$ be arbitrary and write $x^* = \sum_{i=1}^\infty a_i f_i^* + a_\infty S_\infty^*$ as in (3.3). Fix $k \in \mathbf{N}$ and consider

$$I = \{j \le k : |a_i| \ge \delta \text{ for some } i \in [n_j, n_{j+1})\}$$

(note that I depends on x^*, k and δ). Write $I = \{j_1, \ldots, j_r\}$ and pick $p_s, q_s \in [n_{j_s}, n_{j_s+1})$ such that $|a_{p_s}| \ge \delta$ and $|a_{q_s}| < \delta/2$ for $s = 1,\ldots,r$. Since $|a_{q_s} - a_{p_s}| > \delta/2$ for each $s = 1,\ldots,r$, it follows from Lemma 3.1.(ii) that $r = |I| \le 4C/\delta^2$ for some uniform constant $C < \infty$. For $j \notin I$ we have $|a_s| < \delta$ for all $s \in [n_j, n_{j+1})$, so that $|B_{[n_j,n_{j+1})}(x^*)| < \delta$. We get the estimates

$$\begin{aligned}|B_{(n_1,\ldots,n_{k+1})}(x^*)| &\le \frac{1}{k}\Big(\sum_{j\in I}|B_{[n_j,n_{j+1})}(x^*)| + \sum_{j\notin I}|B_{[n_j,n_{j+1})}(x^*)|\Big) \le \frac{2|I|}{k} + \frac{(k-|I|)\delta}{k}\\ &\le \frac{8C}{k\delta^2} + \delta < 2\delta\end{aligned}$$

for all large enough $k = k(\delta)$.

Step 2. Fix a decreasing positive sequence (ε_j) such that $\sum_{j=1}^\infty \varepsilon_j < \varepsilon$. Next apply Claim 1 successively to get a sequence of finite subdivisions $1 = n_{r_1} < n_{r_1+1} < \ldots < n_{r_2} < n_{r_2+1} < \ldots < n_{r_3} < \ldots$ so that

$$|B_{(n_{r_j},\ldots,n_{r_{j+1}})}(x^*)| < \varepsilon_j \text{ for all } x^* \in D \text{ and } j \in \mathbf{N}. \tag{3.4}$$

Put $B_j = B_{(n_{r_j},\ldots,n_{r_{j+1}})}$ and $I_j = [n_{r_j}, n_{r_{j+1}})$ for $j \in \mathbf{N}$. Define the linear map V on J^* by

$$Vx^* = V\Big(\sum_{i=1}^\infty a_i f_i^* + a_\infty S_\infty^*\Big) = x^* - \sum_{j=1}^\infty B_j(x^*) I_j^*, \quad x^* \in J^*.$$

We next verify that V is a weakly compact operator on J^* that satisfies (1.1) for D and the given $\varepsilon > 0$. The uniform bound for $\|V\|$ will require additional tools compared to the argument in section 2.

Claim 2. $\|x^* - Vx^*\| < \varepsilon$ *for all* $x^* \in D$.

Proof of Claim 2. Recall that $\|I_j^*\| = 1$ for $j \in \mathbf{N}$ by (3.1). Hence it follows from (3.4) that

$$\|x^* - Vx^*\| \leq \sum_{j=1}^{\infty} |B_j(x^*)| \cdot \|I_j^*\| < \sum_{j=1}^{\infty} \varepsilon_j < \varepsilon, \quad x^* \in D.$$

Claim 3. $\|V\| \leq 7$ *(independently of the subdivisions).*

Proof of Claim 3. Let U be the linear map

$$Ux^* = \sum_{j=1}^{\infty} B_j(x^*) I_j^*, \quad x^* \in J^*.$$

We will write $x^* = \sum_{s=1}^{\infty} x^*(s) f_s^* + x_\infty^* S_\infty^* \in J^*$ with respect to the Schauder basis $\{f_j^* : j \in \mathbf{N}\} \cup \{S_\infty^*\}$ of J^*. Here $x_\infty^* = \lim_{s\to\infty} b_s = \lim_{s\to\infty} x^*(f_s)$ in terms of the w^*-representation $x^* = (w^*) \sum_{s=1}^{\infty} b_s f_s^* \in J^*$. Thus $|x_\infty^*| \leq \|x^*\|$, so that

$$\|\sum_{s=1}^{\infty} x^*(s) f_s^*\| \leq \|x^*\| + |x_\infty^*| \cdot \|S_\infty^*\| \leq 2\|x^*\| \tag{3.5}$$

for $x^* \in J^*$. We next establish that

$$\|Ux^*\| \leq 3 \tag{3.6}$$

for all finitely supported $x^* \in [f_1^*, \ldots, f_n^*] \cap B_{J^*}$, $n \in \mathbf{N}$. Since $[f_1^*, \ldots, f_n^*] \cap B_{J^*}$ is the closed convex hull of its extreme points, it is enough to show that (3.6) holds for the extreme points x^* of $[f_1^*, \ldots, f_n^*] \cap B_{J^*}$. Towards this it suffices by Proposition 3.4 below to restrict attention to functionals x^* having the special form

$$x^* = \sum_{j=1}^{r} c_j S_j^*, \quad \text{where } \sum_{j=1}^{r} |c_j|^2 = 1 \text{ and } S_1 < \ldots < S_r$$

are intervals of $\mathbf{N}$ with $\max S_r \leq n$. Here $n \in \mathbf{N}$ is arbitrary. We fix some more notation for convenience. Put $I_{j,k} = [n_{r_j+k}, n_{r_j+k+1})$ for $k = 0, \ldots, r_{j+1} - r_j - 1$ corresponding to the subdivision $n_{r_j} < n_{r_j+1} < \ldots < n_{r_{j+1}-1} < n_{r_{j+1}}$ of $I_j = [n_{r_j}, n_{r_{j+1}})$ for $j \in \mathbf{N}$. Thus $\mathbf{N} = \bigcup_{j=1}^{\infty} (\cup_{k=0}^{r_{j+1}-r_j-1} I_{j,k})$ is a partition of $\mathbf{N}$.

Let $x^* = \sum_{i=1}^{r} c_i S_i^*$ be as above. We have

$$Ux^* = \sum_{j=1}^{\infty} B_j(x^*) I_j^* = \sum_{j=1}^{\infty} (\sum_{i=1}^{r} c_i B_j(S_i^*)) I_j^*$$

by definition. Note first that $B_j(S_i^*) = 1$ if the interval $I_j = [n_{r_j}, n_{r_{j+1}}) \subset S_i$. Let A consist of the indices $i \in \{1, \ldots, r\}$ such that $I_j \subset S_i$ for some $j \in \mathbf{N}$, and put $\overline{S_i} \equiv \bigcup_{j: I_j \subset S_i} I_j$ for $i \in A$. Note that $\overline{S_i}$ is an interval contained in S_i. Hence the corresponding coordinates satisfy $Ux^*(s) = c_i$ for $s \in \overline{S_i}$ and $i \in A$.

Suppose next that $I_j = [n_{r_j}, n_{r_{j+1}})$ is not contained in any of the intervals $S_1, \ldots, S_r$. For these coordinates Ux^* looks like

$$(\sum_{i=1}^{r} c_i B_j(S_i^*)) I_j^* \equiv d_j I_j^*. \tag{3.7}$$

Here $B_j(S_i^*) = \frac{1}{n_{r_{j+1}} - n_{r_j}} \sum_{k=0}^{r_{j+1} - r_j - 1} \frac{|I_{j,k} \cap S_i|}{|I_{j,k}|} \in [0, 1)$. Note that $d_j = \sum_{i=1}^{r} B_j(S_i^*) c_i$ is an "absolutely convex" combination of $c_1, \ldots, c_r$, since $\sum_{i=1}^{r} B_j(S_i^*) \le 1$. Put

$$E = \{j \in \mathbf{N} : I_j \nsubseteq S_i \text{ for any } i = 1, \ldots, r\}.$$

Consequently we may coordinatewise split

$$Ux^* = \sum_{i \in A} c_i (\overline{S_i})^* + \sum_{j \in E_1} d_j I_j^* + \sum_{j \in E_2} d_j I_j^* \equiv \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where the above sums are actually finite. Here E_1 and E_2 contain every second index of E, respectively. It is immediate from (3.2) that $\|\Sigma_1\| \le (\sum_{i \in A} |c_i|^2)^{1/2} \le 1$.

Suppose that $j \in E_1$. By (3.7) one has $|d_j| \le |c_{i(j)}| \equiv \max\{|c_i| : I_j \cap S_i \neq \emptyset\}$ for a suitable $i(j) \in \{1, \ldots, r\}$ with $I_j \cap S_{i(j)} \neq \emptyset$. Since E_1 contains every second index of E it follows that $i(j_1) \neq i(j_2)$ once $j_1, j_2 \in E_1$ and $j_1 \neq j_2$. Thus one gets from (3.2) that

$$\|\Sigma_2\| \le (\sum_{j \in E_1} |d_j|^2)^{1/2} \le (\sum_{i=1}^{r} |c_i|^2)^{1/2} = 1.$$

In a similar manner one checks that $\|\Sigma_3\| \le 1$. By putting these estimates together we get $\|Ux^*\| \le 3$ for these particular functionals $x^* = \sum_{j=1}^{r} c_j S_j^* \in [f_1^*, \ldots, f_n^*] \cap B_{J^*}$, which yields (3.6) (modulo Proposition 3.4 below).

Finally, from (3.5), (3.6) and $U(S_\infty^*) = 0$ we obtain by approximation that

$$\|Uy^*\| = \|U(\sum_{s=1}^{\infty} y^*(s) f_s^*)\| \le 3 \|\sum_{s=1}^{\infty} y^*(s) f_s^*\| \le 6 \|y^*\|$$

for $y^* = \sum_{s=1}^{\infty} y^*(s) f_s^* + y_\infty^* S_\infty^* \in J^*$. We deduce that $\|V\| = \|I - U\| \le 7$.

Claim 4. $V \in W(J^*)$.

Proof of Claim 4. Let $(x_n^*) \subset B_{J^*}$ be an arbitrary sequence. We are required to find a subsequence, still denoted by (x_n^*), so that (Vx_n^*) is weakly convergent. Write $x_n^* = \sum_{j=1}^{\infty} a_j^{(n)} f_j^* + a_\infty^{(n)} S_\infty^*$ as in (3.3) for $n \in \mathbf{N}$. By the w^*-sequential compactness and a

diagonalization we may pass to a subsequence and assume without loss of generality that $\lim_{n\to\infty} a_j^{(n)} = a_j$ for $j \in \mathbf{N} \cup \{\infty\}$, $\lim_{j\to\infty} a_j = a$, and

$$x_n^* \xrightarrow{w^*} x^* \equiv (w^*) \sum_{j=1}^{\infty} a_j f_j^* + a_\infty S_\infty^* = \sum_{j=1}^{\infty} (a_j - a) f_j^* + (a_\infty + a) S_\infty^* \in J^*$$

as $n \to \infty$. The latter representation is the norm-convergent one as in (3.3). Put

$$y_n^* = x_n^* - a \sum_{j=1}^{n} I_j^* = \sum_{s=1}^{\max I_n} (a_s^{(n)} - a) f_s^* + \sum_{s=\max I_n+1}^{\infty} a_s^{(n)} f_s^* + a_\infty^{(n)} S_\infty^*$$

as a norm-convergent sum for $n \in \mathbf{N}$. Let $S_\infty^{**} = w^* - \lim_{n\to\infty} f_n$ in J^{**}, so that $S_\infty^{**}(S_\infty^*) = 1$ and $S_\infty^{**}(f_j^*) = 0$ for $j \in \mathbf{N}$. Put $g_j = f_j - S_\infty^{**}$ for $j \in \mathbf{N}$. Then $\{g_j : j \in \mathbf{N}\} \cup \{S_\infty^{**}\}$ is a Schauder basis for J^{**}, for which

$$\lim_{n\to\infty} g_j(y_n^*) = \lim_{n\to\infty} (a_j^{(n)} - a) = a_j - a \quad (j \in \mathbf{N}), \ \lim_{n\to\infty} S_\infty^{**}(y_n^*) = \lim_{n\to\infty} a_\infty^{(n)} = a_\infty.$$

This yields that $y_n^* \xrightarrow{w} y^* \equiv \sum_{j=1}^{\infty} (a_j - a) f_j^* + a_\infty S_\infty^*$ (norm-convergent sum) in J^* as $n \to \infty$.

Note further that $V(I_j^*) = I_j^* - \sum_{k=1}^{\infty} B_k(I_j^*) I_k^* = 0$ for $j \in \mathbf{N}$, since $B_k(I_j^*) = \delta_{k,j}$ for $j, k \in \mathbf{N}$ by definition. Thus $V x_n^* = V y_n^* \xrightarrow{w} V y^*$ in J^* as $n \to \infty$. The proof of Theorem 3.3 will be complete once we have verified the following auxiliary fact that was used in the proof of Claim 3. $\square$

Proposition 3.4. Put $J_n^* = [f_1^*, \ldots, f_n^*] \subset J^*$ *for* $n \in \mathbf{N}$. *Then the extreme points of* $B_{J_n^*}$ *are contained in the set of elements of the form*

$$\sum_{j=1}^{k} c_j S_j^*, \quad \text{where } \sum_{j=1}^{k} |c_j|^2 = 1 \text{ and } S_1 < \ldots < S_k$$

are intervals of $\mathbf{N}$ *with* $\max S_k \leq n$.

Proof. It follows from (3.1) and (3.2) that

$$D = \{\sum_{j=1}^{k} c_j S_j^* : S_1 < \ldots < S_k \text{ intervals, } \max S_k \leq n \text{ and } \sum_{j=1}^{k} |c_j|^2 = 1\} \subset B_{J_n^*}$$

is a symmetric 1-norming set for $J_n = [f_1, \ldots, f_n] \subset J$. Indeed, it is immediate from (3.2) that $|(\sum_{j=1}^{k} c_j S_j^*)(x)| \leq \|x\|$ for $\sum_{j=1}^{k} c_j S_j^* \in D$ and $x \in J_n$. Conversely, for any non-zero $x \in J_n$ there are intervals $S_1 < \ldots < S_k$ with $\max S_k \leq n$, so that $\|x\|^2 = \sum_{j=1}^{k} S_j^*(x)^2$. By choosing $c_j = \|x\|^{-1} S_j^*(x)$ for $j = 1, \ldots, k$ we get that

$$\|x\| = \sum_{j=1}^{k} c_j S_j^*(x) = (\sum_{j=1}^{k} c_j S_j^*)(x),$$

where $\sum_{j=1}^{k} |c_j|^2 = 1$.

Observe next that $\overline{co}(D) = B_{J_n^*}$. In fact, if $\overline{co}(D) \subsetneq B_{J_n^*}$, then the Hahn-Banach theorem would give $x_0^* \in B_{J_n^*}$ and $x \in B_{J_n}$ so that $x_0^*(x) = 1$ and $x^*(x) \leq \alpha < 1 = \|x\|$ for all $x^* \in \overline{co}(D)$. This contradicts the 1-norming property of D. Finally, since $B_{J_n^*} = \overline{co}(D)$, Milman's "converse" to the Krein-Milman theorem (see e.g. [Ph,Prop. 1.5]) yields that the set of extreme points $ext(B_{J_n^*}) \subset D$. $\square$

Proposition 2.5.(i) and Theorem 3.3 have the following consequence.

Corollary 3.5. *$W(J^*)$ has a B.L.A.I.*

Remarks 3.6. (i) Recall that $J^{**} \approx J$, so that $J^{***} \approx J^*$. Hence Theorems 2.2 and 3.3 imply that the k:th dual $J^{(k)}$ has the W.A.P. for all $k \geq 2$. Moreover, $J = [f_n^* : n \in \mathbf{N}]^*$, since (f_n) is a monotone boundedly complete basis for J. It follows that the predual $J_* \equiv [f_n^* : n \in \mathbf{N}]$ of J also has the W.A.P., since $J^* \approx J_* \oplus [S_\infty^*]$ and the W.A.P. is inherited by complemented subspaces (cf. Lemma 5.2.(i) below).

(ii) The averaging functionals $B_j \in J^{**}$ employed in the proof of Theorem 3.3 are not w^*-continuous on J^*, so that the approximating operators $V \in W(J^*)$ are not adjoints of operators on J. Hence Corollary 3.5 does not (by itself) imply the earlier result of [LW,Cor. 2.4] that $W(J)$ has a B.R.A.I.

4. **A family of James-type spaces having the W.A.P.**

The class of quasi-reflexive Banach spaces is extensive, and sections 2 and 3 suggested the question whether there are quasi-reflexive spaces E that *fail* to have the W.A.P. During the course of this work Argyros and Tolias discovered that a class of hereditarily indecomposable (H.I.) spaces constructed recently (for different purposes) in [ArT] contains quasi-reflexive spaces of this kind. We refer to [ArT] for the description of these spaces, and to [ArT,Prop. 14.10] for the details of the following example.

Example 4.1. There is a quasi-reflexive H.I. space E that fails to have the W.A.P.

One reason for such spaces E to fail the W.A.P. appears to be that they admit "few" weakly compact operators in the sense that

$$L(E) = \{\lambda I + V : \lambda \in \mathbf{C}, V \text{ is strictly singular and weakly compact}\}.$$

In contrast the quasi-reflexive spaces studied in this paper have many weakly compact subsets, but they are also sufficiently rich in weakly compact operators to suggest that they may have the W.A.P. The purpose of this section is to extend the results of section 2 to a general class of quasi-reflexive spaces considered by Bellenot, Haydon and Odell [BHO]. This class also contains J, but the reader is expected to already be familiar with the argument from section 2. The desired approximating operators will also be somewhat more involved in the general case.

Let (h_j) be a normalized Schauder basis for a reflexive space E. The Banach space $J(h_j)$ consists of the scalar sequences (a_j) so that $\lim_{j\to\infty} a_j = 0$ and

$$\|(a_j)\| = \sup\{\|\sum_{j=1}^{n}(a_{p_j} - a_{q_j})h_{p_j}\| : 1 \leq p_1 < q_1 < \ldots < p_n < q_n, n \in \mathbf{N}\} < \infty. \quad (4.1)$$

We obtain J with an equivalent norm to (2.1), if (h_j) is the standard coordinate basis of ℓ^2. The reference [BHO] contains the basic information about this construction, where it is discussed in terms the boundedly complete basis (analogous to (3.1) for J). All these spaces $J(h_j)$ are quasi-reflexive of order 1 by [BHO,Thm. 4.1]. In addition, $J(h_j) \approx J(u_j) \approx J(g_j)$ [BHO,Prop. 1.1], where (u_j) is the *unconditionalization* of (h_j) defined by

$$\|\sum_{j=1}^{\infty} a_j u_j\| = \sup\{\|\sum_{j=1}^{\infty} \theta_j a_j h_j\| : (\theta_j) \in \{-1,1\}^{\mathbf{N}}\} < \infty,$$

and (g_j) is the *right dominant* version of (h_j) (or of (u_j)) given by

$$\|\sum_{j=1}^{\infty} a_j g_j\| = \sup\{\|\sum_{j=1}^{\infty} a_{n(i)} h_{m(i)}\| : 1 \leq m(1) \leq n(1) < m(2) \leq n(2) < \ldots\} < \infty.$$

Hence we may and will assume in the sequel that the original basis (h_j) of E is 1-unconditional, and in view of [BHO,Prop. 1.1.(4)] that

$$\|\sum_{i=1}^{\infty} a_{n(i)} h_{m(i)}\| \leq 2\|\sum_{i=1}^{\infty} a_{n(i)} h_{n(i)}\| \tag{4.2}$$

for all $\sum_{j=1}^{\infty} a_j h_j \in E$ and all sequences $1 \leq m(1) \leq n(1) < \ldots < m(i) \leq n(i) < \ldots$.

Let (e_j) stand for the unit vector basis in $J(h_j)$, which is a normalized monotone Schauder basis for $J(h_j)$. Recall that the basic sequence (x_j) in $J(h_j)$ is a *skipped block* basic sequence of (e_j) if for all j there is $n(j) \in \mathbf{N}$ so that $\max supp(x_j) < n(j) < \min supp(x_{j+1})$. Here, as well as in the sequel, the support $supp(x)$ of $x \in J(h_j)$ is with respect to the basis (e_j). Every normalized skipped block basic sequence (x_n) of (e_j) is C-unconditional in $J(h_j)$ with a uniform constant $C < \infty$, see [BHO,Prop. 2.1.(2)].

We will require some additional tools and auxiliary results in order to extend the argument of section 2 to the present setting. Our first result concerns the uniform unconditionality of skipped block sequences of a special type in $J(h_j)$. It is convenient to denote the natural projection of $J(h_j)$ onto the span $[e_s : m \leq s < n]$ by $P_{[m,n)}$, that is, $P_{[m,n)} = P_{n-1}(I - P_{m-1})$ for $2 \leq m < n$. Here (P_n) are the basis projections on $J(h_j)$ with respect to (e_j). Thus $\|P_{[m,n)}\| \leq 2$ for $m \geq 2$.

Lemma 4.2. Let (x_j) be a skipped block basic sequence of (e_j) and let $S_j \subset \mathbf{N}$ be the smallest intervals satisfying $S_j \supset supp(x_j)$ for $j \in \mathbf{N}$. Suppose that $(n_j) \subset \mathbf{N}$ is an increasing sequence so that $n_0 = 1$ and $n_j \in S_{j+1}$ for $j \in \mathbf{N}$, and put

$$y_j = P_{[n_{j-1},n_j)}(x_j + x_{j+1}), \quad j \in \mathbf{N}.$$

Then there is an absolute constant $C < \infty$ so that

$$\|\sum_{j \in B} a_j y_j\| \leq C\|\sum_{j=1}^{\infty} a_j y_j\| \tag{4.3}$$

for all subsets $B \subset \mathbf{N}$ and all norm convergent $\sum_{j=1}^\infty a_j y_j \in J(h_i)$.

Proof. Let $B \subset \mathbf{N}$ be a given set. By approximation it is enough to establish (4.3) with a uniform constant C for all finitely supported sums $\sum_j a_j y_j$. Put $z = \sum_{j \in B} a_j y_j$ and $\tilde{z} = \sum_j a_j y_j$. Suppose that $p_1 < q_1 < \ldots < p_m < q_m$ is a sequence of coordinates norming z, so that

$$\|z\| = \|\sum_{j=1}^m (z(p_j) - z(q_j)) h_{p_j}\| = \|\sum_{j=1}^m b_j h_{p_j}\|,$$

where we put $b_j = z(p_j) - z(q_j) \neq 0$ for $j = 1, \ldots, m$. Here we use the convenient notation $y = \sum_{s=1}^\infty y(s) e_s$ for elements $y \in J(h_j)$.

Let $T_r \subset \mathbf{N}$ be the smallest interval satisfying $T_r \supset supp(y_r)$ for $r \in \mathbf{N}$. We put $E = \{j \le m : \ p_j, q_j \in T_r \text{ for some } r\}$ and $F = \{j \le m : \ p_j \in T_r, q_j \in T_s \text{ for } r < s\}$. We initially split

$$\|\sum_{j=1}^m b_j h_{p_j}\| \le \|\sum_{j \in E} b_j h_{p_j}\| + \|\sum_{j \in F_1} b_j h_{p_j}\| + \|\sum_{j \in F_2} b_j h_{p_j}\| \equiv \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{4.4}$$

Here $F = F_1 \cup F_2$ is the partition of F into every second index. Observe first that $\tilde{b}_j \equiv \tilde{z}(p_j) - \tilde{z}(q_j) = z(p_j) - z(q_j) = b_j$ whenever $j \in E$, so that $\Sigma_1 \le \|\tilde{z}\|$ by definition.

We next verify that $\Sigma_2 \le 2\|\tilde{z}\|$. Let $j \in F_1$. There are three cases to consider:

$$(i)\ p_j \in T_r,\ q_j \in T_s, \quad (ii)\ p_j \in T_s,\ q_j \in T_t, \quad (iii)\ p_j \in T_r,\ q_j \in T_t, \tag{4.5}$$

where $r < s < t$, $r, t \in B$ and $s \notin B$. In case (i) from (4.5) one has $b_j = z(p_j) - z(q_j) = z(p_j)$. Since (x_s) is a skipped block sequence, and $y_s = P_{[n_{s-1}, n_s)}(x_s + x_{s+1})$, there is some index $r_j \in T_s$ for which $y_s(r_j) = 0$. Hence we may move q_j to r_j, so that the difference

$$b_j = z(p_j) = \tilde{z}(p_j) - \tilde{z}(r_j) \equiv \tilde{b}_j$$

can still be used towards computing $\|\tilde{z}\|$ as in definition (4.1). For case (ii) from (4.5) observe first that $b_j = z(p_j) - z(q_j) = -z(q_j)$. There are two possibilities to consider. If $|\tilde{z}(p_j) - \tilde{z}(q_j)| > \frac{|b_j|}{2}$, then we keep the coordinates $p_j < q_j$ towards computing $\|\tilde{z}\|$ as in (4.1). In the opposite case, where $|\tilde{z}(p_j) - \tilde{z}(q_j)| \le \frac{|b_j|}{2}$, we move the coordinate q_j to some $r_j \in T_t$ satisfying $y_t(r_j) = 0$. This implies that

$$|\tilde{z}(p_j) - \tilde{z}(r_j)| = |\tilde{z}(p_j)| \ge |z(q_j)| - |z(q_j) - \tilde{z}(q_j)| = |b_j| - |\tilde{z}(q_j) - \tilde{z}(q_j)| \ge \frac{|b_j|}{2}.$$

Finally, in case (iii) we have $z(p_j) - z(q_j) = \tilde{z}(p_j) - \tilde{z}(q_j)$ (because $r, t \in B$), and we retain the pair $p_j < q_j$.

We get in all cases from (4.5) that $|b_j| \le 2|\tilde{b}_j|$, where $\tilde{b}_j \equiv \tilde{z}(p_j) - \tilde{z}(\tilde{q}_j)$ and $\tilde{q}_j$ stands for either q_j or r_j, depending on the indicated choices. The 1-unconditionality of the basis (h_i) in E yields then that

$$\Sigma_2 = \|\sum_{j \in F_1} b_j h_{p_j}\| \le 2\|\sum_{j \in F_1} \tilde{b}_j h_{p_j}\| \le 2\|\tilde{z}\|,$$

since the sequence of pairs $p_j < \tilde{q}_j$ with $j \in F_1$ are admissible coordinates towards computing $\|\tilde{z}\|$ as in (4.1). Indeed, the fact that F_1 contains every second index of F ensures that the order is preserved in the new sequence if (some of) the coordinates q_j are moved.

The estimate $\Sigma_3 \leq 2\|\tilde{z}\|$ is similar. This completes the proof of (4.3). $\square$

The following combinatorial lemma due to Ptak [Pt] (see also [BHO] for the present formulation) will be a crucial tool towards building certain convex combinations of averaging functionals, which will replace the double averages used in sections 2 and 3. It is convenient to put $(\alpha_1, \ldots, \alpha_k) \in (S_{\ell_1^k})_+$ provided $\sum_{r=1}^k \alpha_r = 1$ and $\alpha_r \geq 0$ for $r = 1, \ldots, k$.

Lemma 4.3. [BHO,Lemma 3.1] *Let $0 < \delta < 1$ be fixed. Suppose that $\mathcal{F}$ is a collection of non-empty finite subsets of $\mathbf{N}$ satisfying the following properties:*

(i) $B \in \mathcal{F}$ whenever $\emptyset \neq B \subset A$ and $A \in \mathcal{F}$.

(ii) For every $k \in \mathbf{N}$ and every convex combination $(\alpha_1, \ldots, \alpha_k) \in (S_{\ell_1^k})_+$, there is $A \in \mathcal{F}$ so that $\sum_{j \in A} \alpha_j \geq \delta$.

Then there is an infinite subset $M \subset \mathbf{N}$ so that $A \in \mathcal{F}$ for all non-empty finite sets $A \subset M$.

We are now ready to prove the main result of this section, which establishes the weakly compact approximation property for the James-like spaces $J(h_j)$. In the proof we will denote $x \in J(h_j)$ by $x = \sum_{s=1}^\infty x(s)e_s$.

Theorem 4.4. Let (h_j) be a normalized Schauder basis for a reflexive Banach space E. Then the quasi-reflexive space $J(h_j)$ has the W.A.P.

Proof. Recall that in view of [BHO,Prop. 1.1] we may assume that the Schauder basis (h_j) for E is 1-unconditional and satisfies the right dominance property (4.2). Let $D \subset B_{J(h_j)}$ be a weakly compact set and let $\varepsilon > 0$. We again split the argument into distinct steps.

Step 1. Let $0 < \delta < 1$ and $n \in \mathbf{N}$ be given. By successive applications of Lemma 2.1 we fix a sequence $n = n_1 < n_2 < \ldots$ in $\mathbf{N}$, so that for every $x = \sum_{s=1}^\infty x(s)e_s \in D$ and $j \in \mathbf{N}$ there is some index $s_j \in [n_j, n_{j+1})$ for which $|x(s_j)| < \frac{\delta}{2^{j+3}}$.

For technical purposes we need to improve this fact by a slight perturbation.

Claim 1. For every $x \in D$ there is a perturbation $x \approx x_0 + \sum_{j=1}^\infty a_j y_j$ satisfying the following properties:

(i) $x_0 = P_{[1,n_1)}(x)$, $y_j \in [e_s : n_j \leq s < n_{j+1}]$, $\|y_j\| = 1$ and $a_j \geq 0$ for $j \in \mathbf{N}$,

(ii) $\|x - (x_0 + \sum_{j=1}^\infty a_j y_j)\| < \frac{\delta}{4}$ and $\|\sum_{j=1}^\infty a_j y_j\| < 2 + \frac{\delta}{4} < 3$,

(iii) for every $j \in \mathbf{N}$ one has $y_j(s_j) = 0$ for some $s_j \in [n_j, n_{j+1})$.

Proof of Claim 1. Let $x = \sum_{s=1}^\infty x(s)e_s \in D$. We put $x_0 = P_{[1,n_1)}(x)$ and

$$v_j = \sum_{s \in [n_j, n_{j+1}); s \neq s_j} x(s)e_s \in [e_s : n_j \leq s < n_{j+1}, s \neq s_j], \quad j \in \mathbf{N}.$$

Consider $y = x_0 + \sum_{j=1}^\infty v_j = x_0 + \sum_{j=1}^\infty a_j y_j$, where $a_j = \|v_j\|$ and $y_j = \frac{v_j}{\|v_j\|}$ (with obvious modifications if $v_j = 0$). Let $z_j = P_{[n_j, n_{j+1})}(x)$, so that $\|z_j - v_j\| = |x(s_j)| < \frac{\delta}{2^{j+3}}$

for $j \in \mathbf{N}$. Hence $\|x - y\| \leq \sum_{j=1}^{\infty} \|z_j - v_j\| < \frac{\delta}{4}$. The other conditions from Claim 1 are satisfied by construction.

We note that in Claim 1 we also get that

$$a_j \leq \|P_{[n_j, n_{j+1})}(x)\| + \|z_j - v_j\| < 2 + \frac{\delta}{2^{j+3}} < 3, \quad j \in \mathbf{N}. \tag{4.6}$$

Step 2. Define the averaging functionals A_j on $J(h_i)$ by

$$A_j(x) = \frac{1}{n_{j+1} - n_j} \sum_{s=n_j}^{n_{j+1}-1} x(s) \quad \text{for } x = \sum_{s=1}^{\infty} x(s) e_s \in J(h_j)$$

and $j \in \mathbf{N}$. Thus $\|A_j\| = 1$ for $j \in \mathbf{N}$, since $|x(s)| \leq \|x\|$ for $x \in J(h_j)$ and $s \in \mathbf{N}$. For any given $k \in \mathbf{N}$ and $(\alpha_1, \ldots, \alpha_k) \in (S_{\ell_1^k})_+$ we introduce the convex combination

$$A(x) = \sum_{j=1}^{k} \alpha_j A_j(x), \quad x \in J(h_j). \tag{4.7}$$

By definition $A(x)$ is a convex combination of the coordinates $\{x(s) : s = n_1, \ldots, n_{k+1} - 1\}$ of $x = \sum_{s=1}^{\infty} x(s) e_s \in J(h_j)$, and $\|A\| \leq 1$. Note that A depends on (n_j), k and $(\alpha_1, \ldots, \alpha_k)$, but our notation does not make this explicit for simplicity.

The double averages involved in the weakly compact approximating operators from sections 2 and 3 relied implicitly on the square variation norm for J. Our next aim is to show (see Claim 2 below) that we may choose $k \in \mathbf{N}$ and a convex combination $(\alpha_1, \ldots, \alpha_k) \in (S_{\ell_1^k})_+$, so that the corresponding A from (4.7) satisfies $|A(x)| < \delta$ for every $x \in D$. For this end Lemma 4.3 will be crucial. We formulate the main technical step here as a separate lemma.

Lemma 4.5. There is an integer $k \in \mathbf{N}$ and a convex combination $(\alpha_1, \ldots, \alpha_k) \in (S_{\ell_1^k})_+$ so that for all $x \in D$ and perturbations $x \approx x_0 + \sum_{i=1}^{\infty} a_i y_i$ as in Claim 1, one has

$$\sum_{i=1}^{k} \alpha_i a_i < \frac{\delta}{4}. \tag{4.8}$$

Proof of Lemma 4.5. Assume to the contrary that for every $k \in \mathbf{N}$ and every convex combination $(\alpha_1, \ldots, \alpha_k) \in (S_{\ell_1^k})_+$ we may find an element $x \in D$ and a perturbation $x \approx x_0 + \sum_{j=1}^{\infty} a_j y_j$ satisfying conditions (i) - (iii) of Claim 1 and (4.6), so that

$$\sum_{j=1}^{k} \alpha_j a_j \geq \frac{\delta}{4}. \tag{4.9}$$

We wish to apply the combinatorial Lemma 4.3 to this setting. Let $\mathcal{F}$ be the collection of finite sets

$$A = \{i: \ a_i \geq \frac{\delta}{8}, \text{where } x_0 + \sum_{j=1}^{\infty} a_j y_j \text{ satisfies } (i),(ii),(iii) \text{ of Claim 1}, (4.6),$$
$$\text{and (4.9) holds for some } k \text{ and } (\alpha_1,\ldots,\alpha_k) \in (S_{\ell_1^k})_+\}.$$

The family $\mathcal{F}$ satisfies the conditions of Lemma 4.3. Indeed,

$$3 \cdot \sum_{i \in A} \alpha_i \geq \sum_{i \in A} \alpha_i a_i \geq \frac{\delta}{4} - \sum_{i \in A^c} \alpha_i a_i > \frac{\delta}{8},$$

since $\sum_{i \in A^c} \alpha_i a_i < \frac{\delta}{8}$. Here $A^c = \{1,\ldots,k\} \setminus A$.

Lemma 4.3 yields an infinite set $M = \{m_i : i \in \mathbf{N}\} \subset \mathbf{N}$ for which $A \in \mathcal{F}$ for all finite subsets $A \subset M$. By applying this fact successively to $A_n = \{m_1,\ldots,m_n\}$ for $n \in \mathbf{N}$, we obtain a sequence of elements $z_n = x_0^{(n)} + \sum_{j=1}^{\infty} a_j^{(n)} y_j^{(n)} \in J(h_j)$ satisfying conditions (i) - (iii) of Claim 1 and (4.6), and where further

$$a_{m_i}^{(n)} \geq \frac{\delta}{8} \quad \text{for } i = 1,\ldots,n. \tag{4.10}$$

By a compactness argument we may pass to a subsequence of (z_n), so that $a_i^{(n)} \to a_i$, $y_i^{(n)} \to v_i$ and $x_0^{(n)} \to v$ in norm for every $i \in \mathbf{N}$ as $n \to \infty$. Here the supports satisfy $supp(v) \subset [1,n_1)$, $supp(v_i) \subset [n_i,n_{i+1})$, and by condition (iii) we may further ensure that $v_i(s_i) = 0$ for some $s_i \in [n_i,n_{i+1})$ for $i \in \mathbf{N}$.

Lemma 4.2 implies that the sequence (v_i) is a C-unconditional basic sequence in $J(h_j)$ for some uniform constant C. Hence, by passing to the limit above we get that

$$\|\sum_{i=1}^{m} a_i v_i\| \leq 3C \tag{4.11}$$

for $m \in \mathbf{N}$. It follows from (4.11) that (v_j) cannot be a boundedly complete basis for $[v_i : i \in \mathbf{N}]$, since $a_{m_i} \geq \frac{\delta}{8}$ for $i \in \mathbf{N}$ by (4.10). Hence [LT,1.c.10] implies that the quasi-reflexive space $J(h_j)$ must contain an isomorphic copy of c_0. This contradiction completes the proof of Lemma 4.5.

Claim 2. There is $k \in \mathbf{N}$ and $(\alpha_1,\ldots,\alpha_k) \in (S_{\ell_1^k})_+$ so that the corresponding convex combination $A = \sum_{j=1}^{k} \alpha_j A_j \in J(h_j)^*$ given by (4.7) satisfies

$$|A(x)| < \delta \quad \text{for } x \in D.$$

Proof of Claim 2. Let $x \in D$. According to Step 1 we may fix a perturbation $x \approx x_0 + \sum_{i=1}^{\infty} a_i y_i$ satisfying conditions (i) - (iii) of Claim 1. Since $A_j(x_0) = 0$ by definition, we get that

$$\begin{aligned}|A_j(x)| &\leq |A_j(x_0 + \sum_{i=1}^{\infty} a_i y_i)| + |A_j(x - (x_0 + \sum_{i=1}^{\infty} a_i y_i))| \\ &\leq |a_j A_j(y_j)| + 2\|x - (x_0 + \sum_{i=1}^{\infty} a_i y_i)\| \leq a_j + \frac{\delta}{2}\end{aligned}$$

for $j \in \mathbf{N}$.

Next we use Lemma 4.5 to find $k \in \mathbf{N}$ and $(\alpha_1, \ldots, \alpha_k) \in (S_{\ell_1^k})_+$ so that (4.8) holds. Then the above estimate implies that

$$|A(x)| \leq \sum_{j=1}^{k} \alpha_j |A_j(x)| \leq \sum_{j=1}^{k} \alpha_j a_j + \frac{\delta}{2} \cdot \sum_{j=1}^{k} \alpha_j < \delta.$$

This completes the proof of Claim 2.

Step 3. Fix a positive decreasing sequence (ε_i) so that $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon/2$. By applying Steps 1 and 2 successively we get a partition $\mathbf{N} = \bigcup_{j=1}^{\infty} I_j$ into successive intervals and a sequence of functionals $(V_j) \subset J(h_j)^*$ that satisfy the following properties for $j \in \mathbf{N}$:

(iv) $|V_j(x)| < \varepsilon_j/2$ for $x \in D$,

(v) V_j is a convex combination of the type (4.7) of averages corresponding to some partition of I_j into successive subintervals,

(vi) $V_j(x) = V_j(\sum_{s \in I_j} x(s) e_s)$ for $x = \sum_{s=1}^{\infty} x(s) e_s \in J(h_i)$.

Write $I_j = [t_j, t_{j+1})$ for $j \in \mathbf{N}$, where $t_1 = 1$. We put

$$s_0 = e_1, \quad s_1 = \sum_{s=2}^{t_2 - 1} e_s, \quad s_j = \sum_{s=t_j}^{t_{j+1}-1} e_s \quad \text{for } j \geq 2.$$

Define the linear map $\tilde{V}$ on $J(h_j)$ by

$$\tilde{V} x = \sum_{j=1}^{\infty} V_j(x) s_{j-1} \quad \text{for } x \in J(h_j). \tag{4.12}$$

Note that definition (4.12) introduces an additional left "shift" on $J(h_j)$ compared to the arguments in sections 2 and 3. It is immediate that $V = I - \tilde{V}$ satisfies

$$\|x - Vx\| = \|\sum_{j=1}^{\infty} V_j(x) s_{j-1}\| \leq 2 \sum_{j=1}^{\infty} |V_j(x)| < \varepsilon \quad \text{for every } x \in D.$$

We verify below in Claims 3 and 4 that $V = I - \tilde{V}$ is the desired weakly compact approximating operator on $J(h_j)$. The right dominance property (4.2) will be essential towards getting a uniform bound for $\|\tilde{V}\|$.

Claim 3. $\|V\| \le 5$ *(independently of the subdivisions).*

Proof of Claim 3. We estimate $\|\tilde{V}\|$. Let $x = \sum_{s=1}^{\infty} x(s)e_s \in J(h_j)$ be finitely supported, and suppose that $supp(x) \subset \bigcup_{j=1}^{n} I_j$ for some $n \in \mathbf{N}$. Let $k_1 < l_1 < \ldots < k_r < l_r$ be a sequence of coordinates that norms $\tilde{V}x$ according to (4.1), so that

$$\|\tilde{V}x\| = \|\sum_{j=1}^{r} a_j h_{k_j}\|,$$

where $a_j = \tilde{V}x(k_j) - \tilde{V}x(l_j)$ for $j = 1, \ldots, r$. Since (h_j) is a 1-unconditional basis of E, we may assume that $a_j \neq 0$ for $j = 1, \ldots, r$.

By conditions (v) and (vi) the element $\tilde{V}x \in J(h_j)$ is constant on each interval I_j with $j \ge 2$. In addition, $\tilde{V}x(1) = V_1(x)$ and $\tilde{V}x(s) = V_2(x)$ for $2 \le s < t_2$. Hence we may assume without loss of generality that no pair k_i and l_i belongs to the same interval $supp(s_j)$ for $i = 1, \ldots, r$, and that $r \le n$.

We claim that

$$\|\sum_j a_{2j+1} h_{k_{2j+1}}\| \le 2\|x\|.$$

From condition (vi) we get that

$$a_{2j+1} = \tilde{V}x(k_{2j+1}) - \tilde{V}x(l_{2j+1}) = V_p(x) - V_q(x)$$

for some $k_{2j+1} \le p < q$. By condition (v) we know that $V_p(x)$ is a convex combination of the coordinates $x(s)$ with $s \in I_p$, and similarly for $V_q(x)$ with respect to I_q. Hence there are coordinates $m_{2j+1} \in I_p$ and $n_{2j+1} \in I_q$, so that

$$b_{2j+1} \equiv |x(m_{2j+1}) - x(n_{2j+1})| \ge |a_{2j+1}|$$

for each j. Here $k_1 \le m_1 < k_3 \le m_3 < \ldots$, and it is easy to convince oneself that $m_1 < n_1 < m_3 < n_3 < \ldots$. Thus we get from (4.1), (4.2) and the 1-unconditionality of (h_j) in E that

$$\|\sum_j a_{2j+1} h_{k_{2j+1}}\| \le \|\sum_j |a_{2j+1}| h_{k_{2j+1}}\| \le 2\|\sum_j b_{2j+1} h_{m_{2j+1}}\| \le 2\|x\|.$$

In a similar fashion one has $\|\sum_j a_{2j} h_{k_{2j}}\| \le 2\|x\|$, so that $\|\tilde{V}x\| \le 4\|x\|$. Consequently $\|V\| \le 1 + \|\tilde{V}\| \le 5$, which completes the proof of Claim 3.

Claim 4. $V = I - \tilde{V} \in W(J(h_j))$.

Proof of Claim 4. It suffices to verify that $(V(\sum_{j=0}^{n+1} s_j)) = ((I - \tilde{V})(\sum_{j=0}^{n+1} s_j))$ is a weak-null sequence in $J(h_j)$, since $\sum_{j=0}^{n+1} s_j \xrightarrow{w^*} \mathbf{1}$ as $n \to \infty$ in $J(h_j)^{**}$. Recall for this that $J(h_j)^{**} = J(h_j) \oplus [\mathbf{1}]$, where $\mathbf{1} = (1, 1, 1, \ldots)$, and that an operator $U \in W(J(h_j))$ if and only if $U^{**}(\mathbf{1}) \in J(h_j)$.

Note first that $\tilde{V}(s_{j+1}) = s_j$ for $j \geq 1$, and that $\tilde{V}(s_0 + s_1) = s_0$. It follows that

$$(I - \tilde{V})(\sum_{j=0}^{n} s_j) = s_{n+1} \xrightarrow{w} 0$$

in $J(h_j)$ as $n \to \infty$. Indeed, if (s_{n+1}) is not weakly null in $J(h_j)$, then it would contain a subsequence equivalent to the unit vector basis of ℓ^1 by [LT,1.c.9] (recall that skipped subsequences of (s_{n+1}) are unconditional, see [BHO,Prop. 2.1.(2)]). This would contradict the quasi-reflexivity of $J(h_j)$. The proof of Theorem 4.4 is complete. $\square$

Remark 4.6. We did not consider the problem whether the dual $J(h_j)^*$ always has the W.A.P. in the setting of Theorem 4.4.

5. **Permanence properties and further positive results.**

In this section we state some simple permanence properties for weakly compact approximation, which imply that certain vector-valued sequence spaces such as $\ell^p(\ell^1)$, $\ell^p(J)$ and $\ell^1(\ell^p)$ have the W.A.P for $1 < p < \infty$. These facts will be needed in section 6.

We first recall the following "dual" version of W.A.P., which has some applications of its own, see [T1], [T2]. The Banach space E is said to have the *inner weakly compact approximation property* (inner W.A.P. for short) if there is a constant $C < \infty$ so that

$$\inf\{\|U - UV\| : V \in W(E), \|V\| \leq C\} = 0 \tag{5.1}$$

for any weakly compact operator $U \in W(E, Z)$, where Z is an arbitrary Banach space. The inner W.A.P. is suggested by an analogous operator reformulation of the W.A.P.: E has the W.A.P. if and only if there is $C < \infty$ so that

$$\inf\{\|U - VU\| : V \in W(E), \|V\| \leq C\} = 0$$

for any weakly compact operator $U \in W(Z, E)$, where Z is an arbitrary Banach space. We recall that the duality between the W.A.P. and the inner W.A.P. is incomplete.

Examples 5.1. If E has the inner W.A.P., then E^* has the W.A.P., see [T1,3.4]. The converse does not always hold: the Johnson-Lindenstrauss space JL fails to have the inner W.A.P., but JL^* has the W.A.P., see [T2,1.4]. Moreover, the fact that E has the W.A.P. does not in general imply that E^* has the inner W.A.P. (indeed, ℓ^1 has the W.A.P., but ℓ^∞ does not have the inner W.A.P., see [T1,3.5.(ii)] or Proposition 6.10 below). Note also that c_0 has the inner W.A.P. by [T1,3.5.(ii)], but c_0 does not have the W.A.P.

It is a simple fact that the (inner) W.A.P. is preserved by complementation. We omit the easy arguments.

Lemma 5.2. Suppose that $M \subset E$ is a complemented subspace, and let P be a projection of E onto M.

(i) If E has the W.A.P with constant C, then M has the W.A.P. with constant $\|P\|C$.

(ii) If E has the inner W.A.P with constant C, then M has the inner W.A.P. with constant $\|P\|C$.

Let R be a Banach space having a normalized 1-unconditional Schauder basis (r_j) and suppose that (E_j) is a sequence of Banach spaces. The vector-valued sequence spaces (or the R-direct sums)

$$R(E_j) = \{x = (x_j) : x_j \in E_j \text{ for } j \in \mathbf{N}, \|x\| \equiv \| \sum_{j=1}^{\infty} \|x_j\| \cdot r_j \|_R < \infty\}$$

provide a suitable setting for our permanence results. Special cases include the familiar direct sums $\ell^p(E_j)$ $(1 \leq p < \infty)$ and $c_0(E_j)$. Let $J_k : E_k \to R(E_j)$ denote the inclusion map and P_k the natural norm-1 projection of $R(E_j)$ onto E_k for $k \in \mathbf{N}$.

Proposition 5.3. *Let (E_j) be a sequence of Banach spaces and suppose that R is a reflexive Banach space having a normalized 1-unconditional Schauder basis (r_j). Then*

(i) $R(E_j)$ has the W.A.P. if and only if E_j has the W.A.P. with a uniform constant.

(ii) $R(E_j)$ has the inner W.A.P. if and only if E_j has the inner W.A.P. with a uniform constant.

(iii) $\ell^1(E_j)$ has the W.A.P. if and only if E_j has the W.A.P. with a uniform constant.

(iv) $c_0(E_j)$ has the inner W.A.P. if and only if E_j has the inner W.A.P. with a uniform constant.

Proof. (i) We put $X = R(E_j)$ for simplicity. If X has the W.A.P. with constant C, then the 1-complemented subspace $E_k \subset X$ has the W.A.P. with the same constant C for $k \in \mathbf{N}$ by Lemma 5.2.(i). (The implication "$\Rightarrow$" is checked similarly for parts (ii), (iii) and (iv).)

Conversely, assume that E_j has the W.A.P. with a uniform constant C for all j. Suppose that $D \subset X$ is a weakly compact subset and $\varepsilon > 0$. Since $P_k D \subset E_k$ is weakly compact for $k \in \mathbf{N}$, there is by assumption $V_k \in W(E_k)$ so that

$$\sup_{y \in P_k D} \|y - V_k y\| < \frac{\varepsilon}{2^k} \quad \text{and} \quad \|V_k\| \leq C.$$

Define $V : X \to X$ by $Vx = (V_k x_k)$ for $x = (x_k) \in X$. Clearly

$$\|Vx\| = \| \sum_{k=1}^{\infty} \|V_k x_k\| \cdot r_k \|_R \leq C \| \sum_{k=1}^{\infty} \|x_k\| \cdot r_k \|_R = C\|x\|$$

for $x = (x_k) \in X$ by the 1-unconditionality of (r_k), so that $V \in L(X)$ and $\|V\| \leq C$.

Let $(x^{(m)}) \subset X$ be a bounded sequence, where $x^{(m)} = (x_j^{(m)})_{j \in \mathbf{N}} \in X$ for $m \in \mathbf{N}$. Then $x^{(m)} \xrightarrow{w} x = (x_j)$ in X as $m \to \infty$ if and only if $x_j^{(m)} \xrightarrow{w} x_j$ as $m \to \infty$ in E_j for $j \in \mathbf{N}$ (cf. [L,6.1] for the special case $\ell^p(E)$). The Eberlein-Smulian theorem implies then that a bounded set $A \subset X$ is relatively weakly compact if and only if $P_j A$ is relatively weakly compact in E_j for $j \in \mathbf{N}$. By applying this fact to VB_X we get that $V \in W(X)$,

since $P_j(VB_X) = V_j B_{E_j}$ is relatively weakly compact for each j. Finally, for $x = (x_k) \in D$ we have

$$\|x - Vx\| = \| \sum_{k=1}^{\infty} \|x_k - V_k x_k\| \cdot r_k\|_R \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

(ii) Suppose that E_j has the inner W.A.P. with a uniform constant C for all j. Let $U \in W(X, Z)$ and $\varepsilon > 0$ be given, where $X = R(E_j)$ and Z is some Banach space. Put $U_k = UJ_k \in W(E_k, Z)$ for $k \in \mathbf{N}$. By assumption there is $V_k \in W(E_k)$ satisfying

$$\|U_k - U_k V_k\| < \frac{\varepsilon}{2^k} \quad \text{and} \quad \|V_k\| \leq C$$

for $k \in \mathbf{N}$. Define $V \in L(X)$ by $Vx = (V_k x_k)$ for $x = (x_k) \in X$. One verifies as in part (i) that $\|V\| \leq C$ and $V \in W(X)$.

We next estimate $\|U - UV\| = \|U^* - V^*U^*\|$. Observe that $X^* = R^*(E_j^*)$, where the biorthogonal sequence (r_j^*) to (r_j) is a 1-unconditional Schauder basis for R^*. Here $U^* = (U_j^*) : Z^* \to R^*(E_j^*)$, and we get for $z^* \in B_{Z^*}$ that

$$\|U^*z^* - V^*U^*z^*\| = \| \sum_{k=1}^{\infty} \|U_k^* z^* - V_k^* U_k^* z^*\| \cdot r_k^*\|_{R^*} \leq \varepsilon.$$

Hence $X = R(E_j)$ has the inner W.A.P.

(iii) Let $D \subset \ell^1(E_j)$ be a weakly compact set and $\varepsilon > 0$. Using Lemma 5.4 below we fix $n \in \mathbf{N}$ so that

$$\sup_{x=(x_j)\in D} \Big(\sum_{j=n+1}^{\infty} \|x_j\| \Big) < \frac{\varepsilon}{2}.$$

The assumption gives operators $V_j \in W(E_j)$ for $j = 1, \ldots, n$ satisfying $\|V_j\| \leq C$ and $\|x_j - V_j x_j\| < \frac{\varepsilon}{2^{j+1}}$ for $x = (x_k) \in D$. Define $V \in L(X)$ by $V(x_k) = (V_1 x_1, \ldots, V_n x_n, 0, 0, \ldots)$ for $(x_k) \in X$. Clearly $\|V\| \leq C$ and $V \in W(X)$. For $x = (x_j) \in D$ we get that

$$\|x - Vx\| \leq \sum_{j=1}^{n} \|x_j - V_j x_j\| + \sum_{j=n+1}^{\infty} \|x_j\| \leq \varepsilon.$$

(iv) Put $X = c_0(E_j)$, and suppose that $U = (U_k) \in W(X, Z)$ and $\varepsilon > 0$. Here $U_k = UJ_k$ for $k \in \mathbf{N}$ and $U^* = (U_k^*)$ is a weakly compact operator. By applying Lemma 5.4 (see below) to the relatively weakly compact set $U^*(B_{Z^*}) \subset X^* = \ell^1(E_j^*)$ there is $n \in \mathbf{N}$ so that

$$\sup_{z^* \in B_{Z^*}} \Big(\sum_{j=n+1}^{\infty} \|U_j^* z^*\| \Big) < \frac{\varepsilon}{2}.$$

Fix $V_j \in W(E_j)$ so that $\|U_j - U_j V_j\| < \frac{\varepsilon}{2^{j+1}}$ and $\|V_j\| \leq C$ for $j = 1, \ldots, n$. Define again $V \in W(X)$ by $V(x_k) = (V_1 x_1, \ldots, V_n x_n, 0, 0, \ldots)$ for $(x_k) \in X$, so that $\|V\| \leq C$. For $z^* \in B_{Z^*}$ we get that

$$\|U^*z^* - V^*U^*z^*\| \leq \sum_{j=1}^{n} \|U_j^* z^* - V_j^* U_j^* z^*\| + \sum_{j=n+1}^{\infty} \|U_j^* z^*\| \leq \sum_{j=1}^{n} \|U_j - U_j V_j\| + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus $\|U - UV\| \le \varepsilon$, so that $c_0(E_j)$ has the inner W.A.P. $\square$

The following auxiliary fact was used in the proof of parts (iii) and (iv) of Proposition 5.3. We sketch the argument of this well known result for completeness.

Lemma 5.4. Let (E_j) be a sequence of Banach spaces and suppose that $D \subset \ell^1(E_j)$ is a weakly compact set. Then for any $\delta > 0$ there is $n = n(D, \delta) \in \mathbf{N}$ so that

$$\sup_{y=(y_j)\in D} \Big(\sum_{j=n}^{\infty} \|y_j\|\Big) \le \varepsilon. \tag{5.2}$$

Proof. Suppose to the contrary that (5.2) does not hold: there is a $\delta > 0$ so that for each $n \in \mathbf{N}$ there is some $y = (y_j) \in D$ satisfying $\sum_{j=n}^{\infty} \|y_j\| > \delta$. Hence there are sequences $(p_n) \subset \mathbf{N}$ and $(y^{(n)}) \subset D$ so that $\sum_{j=p_n+1}^{p_{n+1}} \|y_j^{(n)}\| > \delta$ for $n \in \mathbf{N}$. Here $y^{(n)} = (y_j^{(n)})_{j\in\mathbf{N}} \in \ell^1(E_j)$ for $n \in \mathbf{N}$. It is then easy to verify that a subsequence of $(y^{(n)})$ is equivalent to the unit vector basis of ℓ^1 (this will contradict the weak compactness of D). $\square$

The following novel examples of concrete spaces that have the W.A.P. (or the inner W.A.P.) are immediate from Proposition 5.3.

Corollary 5.5. Let $1 < p < \infty$. Then
(i) $\ell^p(\ell^1)$ and $\ell^1(\ell^p)$ have the W.A.P.,
(ii) $\ell^p(c_0)$ and $c_0(\ell^p)$ have the inner W.A.P.,
(iii) $\ell^p(J)$ and $\ell^p(J^*)$ have the W.A.P.

Proof. Recall that ℓ^1 has the W.A.P., and that c_0 has the inner W.A.P. (see [T1,3.5.(ii)]). Part (iii) follows from Theorems 2.2, 3.3 and Proposition 5.3.(i). $\square$

Remark 5.6. The fact that $\ell^2(J)$ has the W.A.P. sheds some further light on a result of [GST]. Let E be a Banach space and define the "residual" operator $R(S) \in L(E^{**}/E)$ by

$$R(S)(x^{**} + E) = S^{**}x^{**} + E \quad \text{for } x^{**} \in E^{**},\ S \in L(E).$$

It is known (cf. [GST,1.4]) that $\|R(S)\| \le \omega(S) \le \|S\|_w$ for $S \in L(E)$, where $\|S\|_w \equiv \mathrm{dist}(S, W(E))$ and $\omega(\cdot)$ is the measure of weak non-compactness (cf. the Introduction).

According to [GST,2.6] there is a sequence $(S_n) \subset L(\ell^2(J))$ so that $\|S_n\|_w = 1$ for all n, but $\|R(S_n)\| \to 0$ as $n \to \infty$. The precise relation between ω and $\|\cdot\|_w$ on $L(\ell^2(J))$ was not resolved in [GST]. Now an inspection of the arguments of Proposition 5.3.(i) and [AT,Thm. 1] reveals that in fact

$$\omega(S) \le \|S\|_w \le 2\,\omega(S), \quad S \in L(\ell^2(J)).$$

Another natural permanence problem, which we did not pursue here, concerns the W.A.P for the Bochner spaces $L^p(E)$.

Problem 5.7. Does $L^p(E) = L^p([0,1];E)$ have the W.A.P (resp., the inner W.A.P.) whenever E has the W.A.P. (resp., the inner W.A.P.) and $1 < p < \infty$? (The cases $p = 1$ and $p = \infty$ are excluded by known facts, cf. Proposition 6.10).

6. James' tree space JT does not have the W.A.P. and related examples.

This section provides concrete answers to various natural questions about the class of spaces having the W.A.P. We first recall a couple of notions. Let $1 \le p < \infty$ be fixed. The Banach space E is *ℓ^p-saturated* if every infinite-dimensional subspace $M \subset E$ contains an isomorphic copy of ℓ^p. The space E is *somewhat reflexive* if every infinite-dimensional subspace $M \subset E$ contains a reflexive infinite-dimensional subspace. (Here "subspace" always means a closed linear subspace.)

The quasi-reflexive H.I. space E from [ArT,Prop. 14.10] (cf. Example 4.1) that fails the W.A.P. yields a striking counterexample to the following question stated in [AT].

Question 6.1. [AT,p. 370] *Suppose that the quotient E^{**}/E is reflexive. Does E have the W.A.P.?*

Since $\ell^2(J)$ has the W.A.P. by Corollary 5.5.(iii), and $\ell^2(J)^{**}/\ell^2(J) = \ell^2(J^{**}/J) = \ell^2$, there are spaces E with the W.A.P. and E^{**}/E reflexive and infinite-dimensional. Theorem 6.5 below yields a concrete space Y without the W.A.P. for which $Y^{**}/Y = \ell^2$.

Our next question addresses another potential extension of the fact that reflexive spaces have the W.AP.

Question 6.2. Suppose that E is a somewhat reflexive space that has the bounded approximation property (B.A.P.). Does E have the W.A.P.?

The answer to Question 6.2 can be deduced from known results (Theorem 6.5 below contains a different, ℓ^2-saturated example).

Example 6.3. Let E be the separable, somewhat reflexive $\mathcal{L}^\infty$-space constructed by Bourgain and Delbaen, see [B,Ch. III] for a description. Then E has the B.A.P., but E does not have the W.A.P. (see Proposition 6.10 below or [AT,Cor. 3]). $\square$

James' tree space JT was introduced by James [J2] as a useful variation of the ideas underlying J, and its properties were further analyzed e.g. by Lindenstrauss and Stegall [LS]. There is a systematic exposition of the properties of JT in [FG, chapter 3]. The fact that J has the W.A.P. (section 2) suggests the following problem.

Question 6.4. Does JT have the W.A.P.?

The main result of this section (Theorem 6.5) establishes that JT does not have the W.A.P., where JT^{**}/JT is a non-separable Hilbert space. We recall the definition of JT and fix some relevant notation. Let

$$\mathcal{T} = \bigcup_{n=0}^{\infty} \{0,1\}^n$$

be the infinite binary tree equipped with the natural partial order. The *nodes* $\alpha \in \mathcal{T}$ satisfy $\alpha = \emptyset$ or $\alpha = (\alpha_1, \ldots, \alpha_n)$ for some $n \in \mathbf{N}$, where $\alpha_j = 0$ or $\alpha_j = 1$ for $j \in \{1, \ldots, n\}$.

The *length* $|\alpha|$ of $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{T}$ is n. Given $\alpha \in \mathcal{T}$ let $f_\alpha : \mathcal{T} \to \mathbf{R}$ be defined by $f_\alpha(\alpha) = 1$ and $f_\alpha(\beta) = 0$ for $\beta \neq \alpha$. James' tree space JT consists of the functions $\sum_\alpha a_\alpha f_\alpha : \mathcal{T} \to \mathbf{R}$ for which the norm

$$\| \sum_\alpha a_\alpha f_\alpha \| = \sup_{k; S_1, \dots, S_k} (\sum_{j=1}^{k} S_j^*(\sum_\alpha a_\alpha f_\alpha)^2)^{1/2} < \infty, \tag{6.1}$$

where the supremum is taken over disjoint segments $S_1, \dots, S_k$ of $\mathcal{T}$ and $k \in \mathbf{N}$. A *segment* $S \subset \mathcal{T}$ has the form $S = \{\gamma \in \mathcal{T} : \alpha \leq \gamma \leq \beta\}$ for given $\alpha, \beta \in \mathcal{T}$ with $\alpha \leq \beta$, and $S^*(\sum_\alpha a_\alpha f_\alpha) = \sum_{\alpha \in S} a_\alpha$ for $\sum_\alpha a_\alpha f_\alpha \in JT$. It is known that $(f_\alpha)_{\alpha \in \mathcal{T}}$ is a monotone boundedly complete basis for JT (ordered by increasing length of the nodes and from "left to right").

A *branch* $B \subset \mathcal{T}$ is a maximal infinite order interval starting at $\emptyset$. A branch B determines the norm-1 functional $S_B^* \in JT^*$ defined by $S_B^*(\sum_\alpha a_\alpha f_\alpha) = \sum_{\alpha \in B} a_\alpha f_\alpha$. Let Γ be the uncountable collection of all branches of $\mathcal{T}$. Then $JT^{**}/JT = \ell^2(\Gamma)$ isometrically, and

$$JT^* = [\{f_\alpha^* : \alpha \in \mathcal{T}\} \cup \{S_B^* : B \in \Gamma],\tag{6.2}$$

see [LS,Thm. 1] or [FG,3.c.3]. Here $(f_\alpha^*)_{\alpha \in \mathcal{T}}$ is the biorthogonal sequence to $(f_\alpha)_{\alpha \in \mathcal{T}}$ in JT^*. Recall further that JT is ℓ^2-saturated, see [J2,Thm.] or [FG,3.a.8].

The following special notation will be convenient. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{T}$, then $\alpha 0 = (\alpha_1, \dots, \alpha_n, 0)$ is the *left successor* (or left concatenation) and $\alpha 1 = (\alpha_1, \dots, \alpha_n, 1)$ the *right successor* of α. For any $n \in \mathbf{N}$ we put

$$\mathcal{T}_n = \{\alpha \in \mathcal{T} : \text{there are at most } n \ 1's \text{ in } \alpha\}.$$

Thus $\alpha \in \mathcal{T}_n$ if the node α contains at most n "right turns". Put $X_n = [f_\alpha : \alpha \in \mathcal{T}_n]$ for $n \in \mathbf{N}$. Note that $X_n \subset JT$ is a 1-complemented subspace, where the restriction $x \mapsto x_{|\mathcal{T}_n}$ defines the natural projection onto X_n. Indeed, if $S_1, \dots, S_k \subset \mathcal{T}$ are disjoint segments, then $S_j \cap \mathcal{T}_n$ are disjoint segments (possibly empty) in $\mathcal{T}_n$ for $j = 1, \dots, k$.

We are ready for our main results about JT. Below parts (i) and (ii) together imply that X_n has the W.A.P. for all $n \in \mathbf{N}$, but where the smallest constant C in (1.1) is proportional to $\sqrt{n}$. Parts (iii) and (iv) are only based on (ii), but (i) will become useful later (see Example 6.8 and Remark 6.9). The space Y in part (iv) is an example, where Y fails to have the W.A.P. and the reflexive quotient Y^{**}/Y is much "smaller" than JT^{**}/JT. The quasi-reflexive space from [ArT,Prop. 14.10] yields an optimal negative answer to Question 6.1 in terms of minimizing $dim(Y^{**}/Y)$, but our examples are easier. They are further witnesses that the quotient E^{**}/E alone does not decide the W.A.P. of E.

Theorem 6.5. Let $X_n = [f_\alpha : \alpha \in \mathcal{T}_n] \subset JT$ be as above. Then the following properties hold.

(i) X_n has the W.A.P. with constant (at most) $3\sqrt{n}$ for $n \in \mathbf{N}$.

(ii) There is a uniform constant $C > 0$ with the following property: for each $n \in \mathbf{N}$ there is a weakly compact set $D_n \subset X_n$ so that if

$$\sup_{x \in D_n} \|x - Vx\| < \frac{1}{10} \quad \text{and } V \in W(X_n), \tag{6.3}$$

then $\|V\| \geq C\sqrt{n}$.

(iii) JT does not have the W.A.P.

(iv) If $Y = \ell^2(X_n)$, then $Y^{**}/Y = \ell^2$ isometrically and Y does not have the W.A.P.

Proof. We say that $B \subset \mathcal{T}$ is a branch *starting at the node* $\alpha \in \mathcal{T}$, if B is a maximal infinite linearly ordered set so that $\gamma \geq \alpha$ for all $\gamma \in B$. It is convenient to fix, for each $n \in \mathbf{N}$, a partition

$$\mathcal{T}_n = \bigcup_{j=1}^{\infty} B_j^{(n)}$$

into disjoint branches, where every $B_j^{(n)} = (\alpha^{(j,n)}, \alpha^{(j,n)}0, \alpha^{(j,n)}00, \ldots)$ is the "always left" branch starting at some node $\alpha^{(j,n)} = (\alpha_1^{(j,n)}, \ldots, \alpha_k^{(j,n)}) \in \mathcal{T}_n$ with $\alpha_k^{(j,n)} = 1$. We may enumerate these branches by requiring that $B_{r+1}^{(n)}$ starts at the first node $\alpha \in \mathcal{T}_n \setminus \bigcup_{j=1}^{r} B_j^{(n)}$ (enumerated according to increasing length and from "left to right").

(i) We will apply Proposition 5.3. Fix $n \in \mathbf{N}$ and put $B_j \equiv B_j^{(n)}$ for notational simplicity as $j \in \mathbf{N}$. Note that $Y_j \equiv [f_\alpha : \alpha \in B_j] = J$ isometrically for $j \in \mathbf{N}$ according to (6.1) and (3.1). Let $P_j \in L(X_n)$ be the natural norm-1 projection onto Y_j corresponding to the restriction $x \mapsto x_{|B_j}$ for $j \in \mathbf{N}$.

Define a linear map $T : X_n \to \ell^2(J)$ by $Tx = (P_j x)$ for $x \in X_n$. It will be enough to show that T is an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| \leq \sqrt{n}$. Indeed, recall that according to Theorem 2.2 and Proposition 5.3.(i) the direct sum $\ell^2(J)$ has the W.A.P. with constant $C \leq 3$ as defined by (1.1). It is then straightforward to check (using the isomorphism T) that X_n has the W.A.P. with some constant $\tilde{C} \leq 3\sqrt{n}$.

We claim that the following estimates hold, where the left-hand inequality of (6.4) states that T is well-defined $X_n \to \ell^2(J)$.

Claim 1. If $x \in X_n$, then

$$\left(\sum_{j=1}^{\infty} \|P_j x\|^2\right)^{1/2} \leq \|x\| \leq \sqrt{n} \cdot \left(\sum_{j=1}^{\infty} \|P_j x\|^2\right)^{1/2}. \tag{6.4}$$

Proof of Claim 1. We may write $x \in X_n$ coordinatewise as $x = \sum_{j=1}^{\infty} P_j x = \sum_{j=1}^{\infty} x_{|B_j}$, since the branches $\{B_j : j \in \mathbf{N}\}$ form a partition of $\mathcal{T}_n$. The left-hand inequality in (6.4) is then obvious by selecting segments that approximately norm each $P_j x$ and which are wholly contained in B_j.

Suppose next that $S_1, \ldots, S_m \subset \mathcal{T}_n$ are disjoint segments. According to (6.1) we must show that

$$\sum_{j=1}^{m} S_j^*(x)^2 \leq n \cdot \sum_{r=1}^{\infty} \|P_r x\|^2$$

for $x \in X_n$. Recall that the nodes $\alpha \in \mathcal{T}_n$ have at most n right turns, so that

$$n(j) \equiv |\{r \in \mathbf{N} : S_j \cap B_r \neq \emptyset\}| \leq n$$

for $j = 1, \ldots, m$. Write the resulting intersected segments as $S_{j,j(1)}, \ldots, S_{j,j(n)}$, where we put $S_{j,j(r)} = \emptyset$ if $n(j) < r \leq n$. We may thus write $S_j^*(x) = \sum_{r=1}^n S_{j,j(r)}^*(x)$ for $x \in X_n$ and $j = 1, \ldots, m$ (observing the convention that $S_{j,j(r)}^*(x) = 0$ if $S_{j,j(r)} = \emptyset$). Here $(\sum_{r=1}^n S_{j,j(r)}^*(x))^2 \leq n \cdot (\sum_{r=1}^n S_{j,j(r)}^*(x)^2)$ by Hölder's inequality. Hence we get from the above that

$$\sum_{j=1}^m S_j^*(x)^2 = \sum_{j=1}^m (\sum_{r=1}^n S_{j,j(r)}^*(x))^2 \leq n \cdot \sum_{j=1}^m (\sum_{r=1}^n S_{j,j(r)}^*(x)^2) \leq n \cdot \sum_{s=1}^\infty \|P_s x\|^2.$$

For the last estimate regroup the finite sum into those of the disjoint segments $\{S_{j,j(r)} : j = 1, \ldots, m,\ r = 1, \ldots, n\}$ that lie inside any given branch B_s for $s \in \mathbf{N}$.

(ii) Let $n \geq 6$ be fixed. For simplicity we put again $B_j \equiv B_j^{(n)}, j \in \mathbf{N}$, for the partition $\mathcal{T}_n = \bigcup_{j=1}^\infty B_j^{(n)}$ that was fixed at the beginning of the proof. Consider the subset

$$D_n = \bigcup_{\alpha \in \mathcal{T}_n} \{f_\alpha - f_{\alpha 0}, f_{\alpha 0} - f_{\alpha 00}, \ldots\} = \bigcup_{j=1}^\infty \{f_\alpha - f_{\alpha 0} : \alpha \in B_j\} \subset X_n.$$

Here the sequences $(f_\alpha - f_{\alpha 0})_{\alpha \in B_j}$ are formed by the consecutive differences along the "always left" branches B_j in $\mathcal{T}_n$ for $j \in \mathbf{N}$.

Claim 2. The set $D_n \cup \{0\}$ is weakly compact in X_n.

Proof of Claim 2. We will verify that any sequence $(x_m) = (f_{\alpha_m} - f_{\alpha_m 0})$ of distinct points from D_n contains a weak-null subsequence (x_{m_k}) in JT.

There is no loss of generality to assume, by applying Ramsey's classical theorem and passing to a subsequence of (x_m), that either

(6.5) all the nodes α_m lie on a single branch B of $\mathcal{T}$, or

(6.6) all the nodes α_m are pairwise incomparable.

If (6.5) holds, then (x_m) is equivalent to a subsequence of the shrinking basis (e_n) of J given by (2.1), and hence it is weakly null (cf. [FG,2.c.10]). If (6.6) holds, then (x_m) is equivalent to the unit vector basis of ℓ^2, and hence it is again weakly null. Thus Claim 2 holds.

Suppose next that $V \in L(X_n)$ is a weakly compact operator satisfying (6.3) for the weakly compact set $D_n \cup \{0\}$. For simplicity, let (f_m) stand for the node basis of a given branch B_r of the partition of $\mathcal{T}_n$. We make a preliminary observation.

Fact. Given $\delta > 0$ there is a sequence of disjointly supported convex blocks (g_j) of (f_m) so that

$$\|Vg_j - Vg_i\| < \delta, \quad i \neq j. \tag{6.7}$$

Here $\|g_j\| = 1$ for $j \in \mathbf{N}$ in view of (6.1).

Indeed, the weak compactness of V yields a subsequence (f_{m_i}) so that $Vf_{m_i} \xrightarrow{w} x \in JT$ as $i \to \infty$. Then Mazur's theorem gives a sequence of disjointly supported convex blocks (g_j) of (f_{m_i}) so that $\|Vg_j - Vg_i\| < \delta$ whenever $i \neq j$.

Let $[t]$ denote the integer part of $t > 0$. We successively apply the preceding Fact to $[n/2]$ "adjacent" branches in $\mathcal{T}_n$, in the manner described below, to get the element

$$x_n = \sum_{j=1}^{[n/2]} (g_{2j} - g_{2j-1}) + \sum_{j=1}^{[n/2]} (f_{\alpha_j} - f_{\alpha_j 0}) \in X_n. \tag{6.8}$$

The differences $g_{2j} - g_{2j-1}$ and $f_{\alpha_j} - f_{\alpha_j 0}$ are successively chosen as follows for $j = 1, \ldots, [n/2]$:

(6.9) g_1 and g_2 are normalized convex blocks on the node basis determined by the "always left" branch B_1, their supports satisfy $\max supp(g_1) < \gamma_1 < \min supp(g_2)$ for some node $\gamma_1 \in B_1$, and $\|Vg_2 - Vg_1\| < \frac{1}{10}$. (Here the support of the convex combinations is with respect to the node basis).

(6.10) $\alpha_1 = \gamma_1 1$ (the right successor of γ_1).

To continue repeat the above procedure by applying (6.7) to the "always left" branch in $\mathcal{T}_n$ starting from the node $\alpha_1 1 = \gamma_1 11$ (the right successor of α_1). A picture will be helpful at this stage. This construction can be performed $[n/2]$ times, since $\mathcal{T}_n$ allows at most n right turns.

Claim 3.

$$\|x_n\| \leq \sqrt{6n} \quad \text{and} \quad \|Vx_n\| \geq \frac{n}{3}. \tag{6.11}$$

Clearly (6.11) yields that $\|V\| \geq \frac{1}{\sqrt{6n}} \cdot \frac{n}{3} = \frac{1}{3\sqrt{6}} \cdot \sqrt{n}$, which gives (ii) with $C = \frac{1}{3\sqrt{6}}$.

Proof of Claim 3. Let $S_1, \ldots, S_m \subset \mathcal{T}_n$ be given disjoint segments. We have to verify that

$$(\sum_{j=1}^{m} S_j^*(x_n)^2)^{1/2} \leq \sqrt{6n}.$$

Note that x_n is a sum of $4[\frac{n}{2}]$ normalized blocks in JT, namely the convex blocks g_i for $i = 1, \ldots, 2[\frac{n}{2}]$, and f_{α_j} and $f_{\alpha_j 0}$ for $j = 1, \ldots, [\frac{n}{2}]$. From the iterative construction of x_n it follows that for each segment S_j there are at most 3 non-empty disjoint segments $S_{j,1}, S_{j,2}, S_{j,3} \subset S_j$ so that

(6.12) $S_j^*(x_n) = \sum_{i=1}^{3} S_{j,i}^*(x_n)$,

(6.13) each $S_{j,i}$ is contained in the smallest segment containing one of the blocks forming x_n.

Let T_s be the smallest segment in $\mathcal{T}_n$ containing $supp(g_s)$ for $s = 1, \ldots, 2[\frac{n}{2}]$. Note that $\sum_{r=1}^{p} U_r^*(x_n)^2 \leq \|g_s\|^2 = 1$ for each s, whenever $U_1, \ldots, U_p$ are disjoint segments contained in T_s. Hence it follows from Hölder's inequality and (6.12),(6.13) that

$$(\sum_{j=1}^{m} S_j^*(x_n)^2)^{1/2} \leq \sqrt{3}\sqrt{4[\frac{n}{2}]} \leq \sqrt{6n}.$$

This yields the first estimate in (6.11).

Note for the second estimate in (6.11) that according to assumption (6.3) one has

$$V(f_{\alpha_j} - f_{\alpha_j 0}) = f_{\alpha_j} - f_{\alpha_j 0} + z_j,$$

where $\|z_j\| < \frac{1}{10}$ for $j = 1, \ldots, [n/2]$, since $f_{\alpha_j} - f_{\alpha_j 0} \in D_n$. Let $S \subset \mathcal{T}$ be a segment so that $\alpha_j \in S$, but its left successor $\alpha_j 0 \notin S$ for $j = 1, \ldots, [n/2]$. Then we get that

$$\|\sum_{j=1}^{[n/2]} (f_{\alpha_j} - f_{\alpha_j 0})\| \geq |S^*(\sum_{j=1}^{[n/2]} (f_{\alpha_j} - f_{\alpha_j 0}))| = [\frac{n}{2}].$$

Since $\|V(g_{2j} - g_{2j-1})\| < \frac{1}{10}$ for $j = 1, \ldots, [n/2]$ by construction, we obtain that

$$\begin{aligned}\|Vx_n\| &= \|\sum_{j=1}^{[n/2]} V(g_{2j} - g_{2j-1}) + \sum_{j=1}^{[n/2]} V(f_{\alpha_j} - f_{\alpha_j 0})\| \\ &\geq \|\sum_{j=1}^{[n/2]} (f_{\alpha_j} - f_{\alpha_j 0})\| - \sum_{j=1}^{[n/2]} \|z_j\| - \sum_{j=1}^{[n/2]} \|V(g_{2j} - g_{2j-1})\| \\ &\geq [\frac{n}{2}] - \frac{n}{2}(\frac{1}{10} + \frac{1}{10}) \geq \frac{n}{3}.\end{aligned}$$

(iii) This fact follows from part (ii) and Lemma 5.2.(i), since $D_n \subset X_n \subset JT$, where X_n is 1-complemented in JT for all $n \in \mathbf{N}$

(iv) The direct sum $Y = \ell^2(X_n)$ does not have the W.A.P. in view of Proposition 5.3.(i), since according to part (ii) the spaces X_n do not have the W.A.P. with a uniform constant. A modification of the corresponding argument for JT in [LS,Thm. 1] (see also [FG,3.c.3]) yields that $X_n^{**}/X_n = \ell^2$ isometrically for all $n \in \mathbf{N}$. Hence $\ell^2(X_n)^{**}/\ell^2(X_n) = \ell^2(X_n^{**}/X_n) = \ell^2$. $\square$

Remark 6.6. Lindenstrauss and Stegall [LS] defined a function space version of J. James' function space JF does not have the W.A.P., since the separable space JF contains a (complemented) copy of c_0, see [LS,p. 95].

The property defined by (1.1) should more precisely be called the *bounded* W.A.P. We say that E has the *unbounded W.A.P.* if for every weakly compact set $D \subset E$ and $\varepsilon > 0$ there is $V \in W(E)$ satisfying

$$\sup_{x \in D} \|x - Vx\| < \varepsilon. \tag{6.14}$$

The reason for our unorthodox terminology is that the known applications of weakly compact approximation are related to the property defined by (1.1), rather than the one by (6.14). Recall that there are spaces that have the (finite rank) approximation property A.P., but not the B.A.P., see [LT,1.e]. This raises another problem.

Question 6.7. Is there a space E that has the unbounded W.A.P., but not the W.A.P.?

It turns out that Theorem 6.5 yields concrete examples of this kind, so that the unbounded W.A.P. is a strictly weaker notion than the W.A.P.

Example 6.8. Let $X_n = [f_\alpha : \alpha \in \mathcal{T}_n] \subset JT$ *be the spaces from Theorem 6.5 for* $n \in \mathbf{N}$*, and let* $Z = \ell^1(X_n)$ *be their direct* ℓ^1*-sum. Then* Z *has the unbounded W.A.P., but not the W.A.P.*

Proof. Proposition 5.3.(iii) yields that $Z = \ell^1(X_n)$ does not have W.A.P., since the spaces X_n do not have the W.A.P. with a uniform constant according to Theorem 6.5.(ii). On the other hand, since X_n has the W.A.P. for all $n \in \mathbf{N}$ by Theorem 6.5.(i), a simple modification of the argument for Proposition 5.3.(iii) implies that $Z = \ell^1(X_n)$ does have the unbounded W.A.P. Indeed, recall that the relevant approximating operators $V \in W(Z)$ were defined by $Vx = (V_1x_1, \ldots, V_nx_n, 0, 0, \ldots)$, $x = (x_k) \in Z$, for suitably chosen $n \in \mathbf{N}$ and $V_j \in W(X_j)$ for $j = 1, \ldots, n$. □

Remark 6.9. The space JT does not even have the unbounded W.A.P. Indeed, let

$$\tilde{D} = \bigcup_{n=1}^{\infty} D_n \cup \{0\} \subset \ell^2(X_n)$$

be the coordinatewise union in the direct ℓ^2-sum, where the weakly compact sets $D_n \subset X_n$ are those of the proof of Theorem 6.5.(ii) for $n \in \mathbf{N}$. The set $\tilde{D}$ is relatively weakly compact in $\ell^2(X_n)$ (cf. the proof of Proposition 5.3.(i)). Note that $\ell^2(X_n) \subset \ell^2(JT)$, where $\ell^2(JT)$ embeds as a complemented subspace of JT, see [FG,3.a.17]. Fix a linear embedding $T : \ell^2(JT) \to JT$, and a projection P of JT onto $T(\ell^2(X_n))$.

Suppose that for any $\varepsilon > 0$ there is $V \in W(JT)$ satisfying $\|x - Vx\| < \varepsilon$ for all $x \in T(\tilde{D})$. It is then easy to check that for every $n \in \mathbf{N}$ there is $V_n \in W(X_n)$, so that

$$\|V_n\| \le C\|V\| \quad \text{and} \quad \sup_{z \in D_n} \|z - V_n z\| < c \cdot \varepsilon,$$

where $C > 0$ and $c > 0$ are uniform constants that only depend on $\|T\|$, $\|T^{-1}\|$ and $\|P\|$. This contradicts Theorem 6.5.(ii) with $\epsilon > 0$ small enough and $n \in \mathbf{N}$ large enough. □

We next state for completeness two simple conditions which guarantee that spaces with the Dunford-Pettis property fail to have the (inner) W.A.P. (see also [AT, Prop. 2] and [T1,3.3]). Recall that a Banach space E has the *Dunford-Pettis property* (DPP) if $\|Vx_n\| \to 0$ as $n \to \infty$ whenever $V \in W(E, F)$ and $(x_n) \subset E$ is a weak-null sequence. The space E has the *Schur property* if $\|x_n\| \to 0$ as $n \to \infty$ for every weak-null sequence $(x_n) \subset E$. The survey [Di] contains a lot of information about the Dunford-Pettis and the Schur properties. The known facts that $L^1(0,1)$ and $L^\infty(0,1)$ have neither the W.A.P. nor the inner W.A.P., as well as many additional examples, can be recovered from the following proposition.

Proposition 6.10. Let E *be a Banach space having the DPP.*

(i) If E *has the W.A.P., then* E *has the Schur property. In particular, if* E *contains an infinite-dimensional reflexive subspace* M*, then* E *fails to have the W.A.P.*

(ii) If E has an infinite-dimensional reflexive quotient space E/M, then E fails to have the inner W.A.P.

Proof. (i) If E does not have the Schur property, then there is a weak-null sequence $(x_n) \subset E$ so that $\|x_n\| \geq c > 0$ for $n \in \mathbf{N}$. Then $\|Vx_n\| \to 0$ as $n \to \infty$ by the Dunford-Pettis property of E for any $V \in W(E)$. Hence

$$\|x_n - Vx_n\| \geq c - \|Vx_n\| \geq \frac{c}{2}$$

for all large enough $n \in \mathbf{N}$, so that E does not have the W.A.P. In particular, if E contains an infinite-dimensional reflexive subspace, then E cannot have the Schur property.

(ii) Let $Q : E \to E/M$ stand for the weakly compact quotient map. Suppose that there is a sequence $(V_n) \subset W(E)$ so that $\|Q - QV_n\| \to 0$ as $n \to \infty$. It follows that QV_n is a compact operator $E \to E/M$ for $n \in \mathbf{N}$, since E has the DPP. Hence the quotient map Q is a compact operator onto E/M, which is not possible. $\Box$

Note that c_0 has the property that every infinite-dimensional subspace $M \subset c_0$ fails to have the W.A.P. This follows from Proposition 6.10.(i) and the fact that c_0 is complementedly c_0-saturated, see [LT,2.a.2]. This fact is another point of difference between the W.A.P. and classical approximation properties.

It is clear that E has the W.A.P. if E has the Schur property and the B.A.P., since $W(E) = K(E)$ in this case. Any space E with the Schur property is ℓ^1-saturated by Rosenthal's ℓ^1-theorem (see [LT,2.e.5]). This fact suggests the following question.

Question 6.11. Suppose that E is an ℓ^1-saturated Banach space that has the B.A.P. Does E have the W.A.P.?

We answer Question 6.11 by showing that the Lorentz sequence spaces $d(w,1)$ fail to have the W.A.P. Let $w = (w_j)$ be a positive non-increasing sequence satisfying

$$w_1 = 1, \quad \lim_{j\to\infty} w_j = 0 \quad \text{and} \quad \sum_{j=1}^{\infty} w_j = \infty. \tag{6.15}$$

Recall that $d(w,1)$ consists of the scalar sequences $x = (x_j)$ for which

$$\|x\| = \sum_{j=1}^{\infty} w_j x_j^* < \infty,$$

where (x_j^*) is the non-increasing rearrangement of $(|x_j|)$. The space $d(w,1)$ is ℓ^1- saturated by [LT,4.e.3], but $d(w,1)$ does not have the DPP, since the coordinate basis (e_n) and its biorthogonal sequence (e_n^*) in $d(w,1)^*$ are weakly null. In place of Proposition 6.10 we will use the (sub)symmetry of the Schauder basis (e_n) for $d(w,1)$.

Let E be a Banach space. Recall that a Schauder basis (e_n) for E is *symmetric* if $(e_{\pi(n)})$ and (e_n) are equivalent for all permutations π of $\mathbf{N}$. The basis (e_n) is *subsymmetric*, if (e_n) is unconditional and (e_{m_n}) is equivalent to (e_n) for all subsequences $m_1 < m_2 < \ldots$.

Every symmetric basis is also subsymmetric [LT,3.a.3]. Let (x_j) and (y_j) be basic sequences in E. Recall that (x_j) is said to *dominate* (y_j) if $\sum_{j=1}^{\infty} c_j y_j$ converges in E whenever $\sum_{j=1}^{\infty} c_j x_j$ converges in E.

Example 6.12. $d(w,1)$ *does not have the W.A.P.*

Proof. The set $D = \{e_n : n \in \mathbf{N}\} \cup \{0\} \subset d(w,1)$ is weakly compact, since (e_n) is a weak-null sequence. We will show that D cannot be approximated in the sense of (1.1). Suppose for this purpose that $V \in L(d(w,1))$ satisfies $\sup_{n\in\mathbf{N}} \|e_n - Ve_n\| < \frac{1}{10}$, and put $x_n = Ve_n$ for $n \in \mathbf{N}$. Then the sequence (x_n) is semi-normalized and weakly null.

Claim. $V \notin W(d(w,1))$.

First choose a basic subsequence (x_{n_j}) so that (x_{n_j}) is equivalent to a block basic sequence (y_j) of (e_n), where $\|x_{n_j} - y_j\| \to 0$ as $j \to \infty$. Put $y_j = \sum_{k=p_j}^{q_j} a_k e_k$ for $j \in \mathbf{N}$, where $p_1 < q_1 < p_2 < q_2 < \ldots$ is a suitable sequence. It is obvious that (e_{n_j}) dominates $(x_{n_j}) = (Ve_{n_j})$. We next verify that (x_{n_j}) dominates (e_{n_j}), so that (x_{n_j}) and (e_{n_j}) will be equivalent basic sequences in $d(w,1)$.

We may assume by approximation that $n_j \in [p_j, q_j]$ and $a_{n_j} = e^*_{n_j}(y_j) > \frac{9}{10}$ for $j \in \mathbf{N}$. Let $c_1, \ldots, c_r$ be scalars and $r \in \mathbf{N}$. It follows from the 1-unconditionality of the basis (e_j) that

$$\|\sum_{j=1}^{r} c_j y_j\| = \|\sum_{j=1}^{r}(\sum_{k=p_j}^{q_j} c_j a_k e_k)\| \geq \frac{9}{10}\|\sum_{j=1}^{r} c_j e_{n_j}\|,$$

and hence (x_{n_j}) dominates (e_{n_j}).

It follows that the restriction of V determines a linear isomorphism $[e_{n_j} : j \in \mathbf{N}] \to [Ve_{n_j} : j \in \mathbf{N}]$, since the sequences (e_{n_j}) and (Ve_{n_j}) are equivalent. Here $[e_{n_j} : j \in \mathbf{N}] \approx d(w,1)$, because (e_n) is a (sub)symmetric basis. This implies the Claim.. □

Remark 6.13. The argument of Example 6.12 actually yields a more general observation, which applies e.g. to certain Orlicz sequence spaces (see Chapter 4 of [LT]):

Suppose that E *is a non-reflexive Banach space which has a weak-null, subsymmetric Schauder basis* (e_n). *Then* E *does not have the W.A.P.*

Azimi and Hagler [AH] introduced a class of spaces that provides a second solution to Question 6.11 (with some additional properties). Let $w = (w_j)$ be a positive non-increasing sequence satisfying (6.15). The Azimi-Hagler space $X(w)$ consists of the scalar sequences $x = (x_j)$ for which

$$\|x\| = \sup_{n; F_1 < \ldots < F_n} \sum_{j=1}^{n} w_j |\sum_{k\in F_j} x_k| < \infty. \tag{6.16}$$

The supremum is taken over all finite intervals $F_1 < \ldots < F_n$ of $\mathbf{N}$ and $n \in \mathbf{N}$. The Banach space $X(w)$ is ℓ^1- saturated, but it does not have the Schur property, see [AH,Thm. 1]. One point of interest in $X(w)$ comes from the facts that the coordinate basis (e_n) is not a subsymmetric basis for $X(w)$ (see the Remark on [AH,p. 295]), and (e_n) does not even contain any weakly convergent subsequences (see [AH,Thm. 1.(3)]). Hence the approach of Example 6.12 must be refined.

Let $P_{m,n}$ denote the natural projection of $X(w)$ onto $[e_s : m \leq s \leq n]$ for $m \leq n$. Thus $\|P_{m,n}\| \leq 2$.

Example 6.14. *$X(w)$ does not have the W.A.P.*

Proof. Put $z_n = e_{2n} - e_{2n-1}$ for $n \in \mathbf{N}$. Then (z_n) is a weak-null sequence in $X(w)$ (see [AH,Lemma 6]), so that $\{z_n : n \in \mathbf{N}\} \cup \{0\}$ is a weakly compact set. Suppose that $V \in L(X(w))$ satisfies

$$\sup_{n\in\mathbf{N}} \|z_n - Vz_n\| < \frac{1}{10},$$

and set $x_n = Vz_n$ for $n \in \mathbf{N}$. Thus (x_n) is a semi-normalized weak-null sequence.

Claim. $V \notin W(X(w))$.

By the standard gliding hump argument we may first choose a subsequence (x_{n_j}) and natural numbers $p_1 < q_1 < p_2 < q_2 < p_3 < \ldots$, so that

(i) (x_{n_j}) and (y_j) are equivalent basic sequences, where $y_j = P_{p_j,q_j}(x_{n_j})$ for $j \in \mathbf{N}$,

(ii) $y_j = u_j + a_j e_{2n_j-1} + b_j e_{2n_j} + v_j$, where $p_j < 2n_j - 1 < 2n_j < q_j$, and the supports satisfy $supp(u_j) \subset [p_j, 2n_j - 1)$ and $supp(v_j) \subset (2n_j, q_j]$ for $j \in \mathbf{N}$.

(iii) $|a_j + 1| < \frac{2}{10}$ and $|b_j - 1| < \frac{2}{10}$ for $j \in \mathbf{N}$.

Property (iii) follows from the fact that $\|y_j - z_{n_j}\| = \|P_{p_j,q_j}(x_{n_j} - z_{n_j})\| < \frac{2}{10}$.

Clearly (z_{n_j}) dominates $(x_{n_j}) = (Vz_{n_j})$. By property (i) it suffices to verify that there is $c > 0$ so that

$$\|\sum_j c_j y_j\| \geq c \cdot \|\sum_j c_j z_{n_j}\| \tag{6.17}$$

for all scalars $c_1, c_2, \ldots$. Indeed, in that event the basic sequences (x_{n_j}) and (z_{n_j}) are equivalent, and the fact that $X(w)$ is ℓ^1-saturated [AH, Thm. 1] will imply that V fixes some ℓ^1-copy contained in $[z_{n_j} : j \in \mathbf{N}]$.

It is enough to verify (6.17) for all finite sums $z = \sum_{j=1}^r c_j z_{n_j}$ and $y = \sum_{j=1}^r c_j y_j$. Put $F^*(x) = \sum_{s\in F} x(s)$ for $x = \sum_{s=1}^\infty x(s)e_s \in X(w)$, whenever $F \subset \mathbf{N}$ is a finite interval. Suppose that $F_1 < F_2 < \ldots < F_m$ are finite intervals for which $\sum_{i=1}^m w_i |F_i^*(z)| = \|z\|$. The non-zero terms $|F_i^*(z)|$ have the form $|c_j|$ or $|c_j - c_k|$ for suitable $j < k$. Indeed, there is no contribution to $F_i^*(z)$ from the terms $c_l(e_{2n_l-1} - e_{2n_l})$, where both $2n_l - 1, 2n_l \in F_i$.

If i is such that $|F_i^*(z)| = |c_j|$, then we may replace F_i by $G_i = \{2n_j - 1\}$ or $G_i = \{2n_j\}$ without affecting $|F_i^*(z)| = |c_j|$. The choice of $2n_j - 1$ or $2n_j$ is according to which of these indices contributes the term $|c_j|$. Thus $G_i^*(y) = b_j c_j$ or $G_i^*(y) = a_j c_j$, so that (iii) yields

$$|G_i^*(y)| \geq \min\{|a_j|, |b_j|\} \cdot |c_j| \geq \frac{8}{10}|c_j| = \frac{8}{10}|F_i^*(z)|. \tag{6.18}$$

If $|F_i^*(z)| = |c_j - c_k|$ for some $j < k$, then we consider two singletons $G_{i,1} < G_{i,2}$ of the preceding type instead of F_i. In this case we get as above that

$$|F_i^*(z)| = |c_j - c_k| \leq |c_j| + |c_k| \leq \frac{10}{8}(|G_{i,1}^*(y)| + |G_{i,2}^*(y)|). \tag{6.19}$$

Put $A = \{i : |F_i^*(z)| = |c_j| \text{ for some } j\}$ and $B = \{i : |F_i^*(z)| = |c_j - c_k| \text{ for some } j < k\}$. We get from (6.18) and (6.19) that

$$\begin{aligned} \|z\| &= \sum_{i\in A} w_i |F_i^*(z)| + \sum_{i\in B} w_i |F_i^*(z)| \\ &\le \frac{10}{8}\Big(\sum_{i\in A} w_i |G_i^*(y)| + \sum_{i\in B} w_i |G_{i,1}^*(y)|\Big) + \frac{10}{8}\sum_{i\in B} w_i |G_{i,2}^*(y)| \le \frac{20}{8}\|y\|. \end{aligned}$$

Here $\sum_{i\in B} w_i |G_{i,2}^*(y)| \le \|y\|$, since the weight sequence (w_i) is non-increasing. □

Remark 6.15. The arguments for Examples 6.12 and 6.14 demonstrate that $d(w,1)$ and $X(w)$ even fail to have the unbounded W.A.P.

The inner W.A.P. (see section 5) is more difficult to study. Our final example shows that the dual JT^* of the James tree space does not have the inner W.A.P. We will require the following facts from [LS,Thm. 1]: JT has a predual B and $B^{**}/B = \ell^2(\Gamma)$ isometrically, where Γ is the uncountable collection of all branches of $\mathcal{T}$. In particular, $JT^{***}/JT^* = \ell^2(\Gamma)$.

Example 6.16. *JT^* does not have the inner WAP.*

Proof. The argument is a modification of that of [T2,1.4] for the Johnson-Lindenstrauss space. Let $q : JT^* = B^{**} \to \ell^2(\Gamma)$ be the weakly compact quotient map. Suppose to the contrary that there is a sequence $(V_n) \subset W(JT^*)$ such that

$$\lim_{n\to\infty} \|q - qV_n\| = 0. \tag{6.20}$$

Recall next that any weakly compact set $D \subset JT^*$ is norm separable, since (D, w) is metrizable in this case. This is based on the fact that JT is a separable space not containing any copies of ℓ^1 (see [LS,Cor. 1] or [FG,3.a.8]), so that JT is w^*-sequentially dense in JT^{**} by the Main Theorem of [OR].

Deduce that the closure $\overline{q(V_n B_{JT^*})}$ is a norm separable set in $\ell^2(\Gamma)$ for $n \in \mathbf{N}$. Thus (6.20) implies that $B_{\ell^2(\Gamma)} = \overline{q(B_{JT^*})}$ is also norm separable by approximation. This contradicts the non-separability of $\ell^2(\Gamma)$. □

If E has the inner W.A.P., then $W(E)$ has a B.R.A.I. by [BD,Prop. 11.2] (cf. also the proof of Proposition 2.5.(i)). Thus Example 6.16 and [LW,Cor. 2.4] suggest

Problems 6.17. (i) Does J have the inner W.A.P.? (ii) Does JT have the inner W.A.P.? (iii) Does JT^* have the W.A.P.?

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Addresses:
(Odell)
Department of Mathematics
The University of Texas at Austin
Austin, TX 78712
USA

e-mail: *odell@math.utexas.edu*

(Tylli)
Department of Mathematics
P.B. 4 (Yliopistonkatu 5)
FIN-00014 University of Helsinki
Finland

e-mail: *hojtylli@cc.helsinki.fi*